Preliminary Examination: Partial Differential Equations, Fall Semester, August 18, 2010

Name: ________________________________

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There are 5 problems. Each problem is worth 25 points. You are required to do 4 of them. Please indicate which 4 you choose—Note: Only 4 problems will be graded. A sheet of convenient formulae is provided.

1. **General Fourier Series and Convergence behavior** (25 points.)

Let

\[ f(x) = \cos(x) + |\cos(x)| \]  

on the interval \([0, 2\pi]\).

(a) (10 points) Determine the Fourier Series of \(f(x)\) and sketch the graph of \(f(x)\).

(b) (10 points) What can be said about the convergence of the Fourier Series for \(f(x)\), e.g. mean convergence, pointwise convergence, uniform convergence? Give a sentence or two justifying your comments.

(c) (5 points) State a theorem involving pointwise convergence of the Fourier Series of \(f'(x)\), the derivative of \(f(x)\). Justify your answer.

2. **Dirichlet’s Theorem(s)**

(a) (5 points) State Dirichlet’s Theorem, including any sufficient conditions on a given function required for the theorem to hold.

(b) (10 points) Does the following expression satisfy Dirichlet’s Theorem: \(\sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k^r}\). Why or why not?

(c) (5 points) What can be said about the Fourier representation of \(f'(x)\), its Fourier coefficients and their decay rates, etc. Be explicit.

(d) (5 points) State Parseval’s Relation, including all the conditions required for the statement to hold.

3. **Wave equation** (25 points.)

\[ u_{tt} = c^2 u_{xx} + \cos(x) \cos(ct), \quad t > 0, \quad -\pi < x < \pi, \]

with periodic boundary conditions in \(x\) subject to the initial condition \(u(x, 0) = 0\), and \(u_t(x, 0) = 3 \cos(2x), -\pi < x < \pi\).

(a) (10 points) Determine \(u(x, t)\) for \(t > 0\) and \(-\pi < x < \pi\).

(b) (7 points) Verify directly that your solution from (a) actually solves the given problem.
(c) (3 points) Is \( u(x,t) \) continuous in \( x \) for \( t > 0 \) and \( -\pi < x < \pi \)? Justify your answer. Is \( u(x,t) \) continuous in \( t \) for \( t > 0, -\pi < x < \pi \)? Justify your answer.

(d) (3 points) Is \( u(x,t) \) periodic in \( x \) for \( t > 0 \) and \( -\pi < x < \pi \)? Justify your answer. Is \( u(x,t) \) periodic in \( t \) for \( t > 0 \) and \( -\pi < x < \pi \)? Justify your answer.

(d) (2 points) Based on your solution from part (a), evaluate and sketch \( u(x,t) \) at \( ct = \pi \) and at \( ct = 2\pi \).

4. Possion’s Equation (25 points.)

Consider Possion’s equation

\[
\nabla^2 u = f
\]

in the rectangular domain \( D = \{0 < x < a, 0 < y < b\} \) and \( u(x, y) = g(x, y) \), for \((x, y) \in \partial D\).

(a) (15 points) State and prove the Maximum Principle for the homogenous problem.

(b) (10 points) Can this problem have more than one regular (\( C^2 \) interior and continuous up to boundary) solution for smooth functions \( f \) and \( g \)? If so, choose specific functions \( f \) and \( g \), and construct two solutions. Be specific. If not, prove uniqueness, including any constraints on \( f \) and \( g \) needed for your proof.

5. Quasi-Linear first order PDE and Method of Characteristics (25 points.)

(a) (10 points) Prove that in order for \( u(x,t) = f(x/t) \) to be a non-constant solution of

\[
\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0
\]

\( f \) must be the inverse of \( a(u) \).

(b) (10 points) Use the method of characteristics to solve the following differential equation

\[
\frac{\partial u}{\partial t} + e^u\frac{\partial u}{\partial x} = 0
\]

for \( x > 0 \) and \( t > 0 \),

and with initial data \( u(x,0) = 2 \), for \( x > 0 \) and \( u(0,t) = 1 \), \( t > 0 \). (Give a complete solution.)

(c) (5 points) Sketch the characteristics of the problem in part (b).