1. Eigenvalue Problem (25 points). Solve the eigenvalue problem

\[ x^2 y'' + 3xy' + \mu y = 0, \quad 1 < x < e, \quad y(1) = y(e) = 0. \]

Assume that \( \mu > 1. \) (Hint: Look for solutions of the form \( y(x) = x^m \) and determine \( \mu \))

2. Fourier Series (25 points).

(a) Find the Fourier cosine series for the function

\[ f(x) = x, \quad 0 \leq x \leq 1. \]

To what function on \( \mathbb{R} \) does this series converge? (Take care to specify the function for each \( x \)).

(b) In certain cases a Fourier series may be differentiated term-by-term to obtain the series for \( f'(x) \). State a theorem to this effect.

(c) Does this theorem apply to (a)? If so, what is the Fourier series for \( f'(x) \)? To what function on \( \mathbb{R} \) does this series converge?

(d) In certain cases a Fourier series may be integrated term-by-term to obtain the Fourier series for \( \int f(x) \, dx \). State AND PROVE a theorem to this effect.

(e) Does this theorem apply to (a)? If so compute the Fourier series for \( \int f(x) \, dx \). To what function on \( \mathbb{R} \) does this series converge?

3. Damped Wave equation (25 points). Consider the PDE

\[ u_{tt} + u_t = u_{xx}, \quad 0 \leq x \leq \pi, \quad t \geq 0, \quad (1) \]

with the boundary conditions \( u(0, t) = u(\pi, t) = 0 \), and initial data \( u(x, 0) = f(x), u_t(x, 0) = 0 \).

(a) Show that any solution of (1) has the property that \( \lim_{t \to \infty} u(x, t) = 0 \).

(b) Find a formal solution for \( u(x, t) \).

(c) Under what conditions on \( f(x) \) is your formal solution a true solution of (1)?

(d) What is the solution when \( f(x) = \sin x \)?
4. First Order PDE (25 points). Let $\varphi(x, y, z)$ solve the PDE

$$x \frac{\partial \varphi}{\partial x} + y \frac{\partial \varphi}{\partial y} + z \frac{\partial \varphi}{\partial z} = \varphi$$

(2)

where $\varphi(x, y) = F(x, y)$ on the surface $z = x^2 + y$.

(a) Find the characteristics of (2).
(b) Find the formal solution of (2).
(c) What do you need to assume about $F(x, y)$ for your solution to solve (2)?
(d) On what region of $\mathbb{R}^3$ is the solution you obtained in (b) valid? What goes wrong and why?

5. An Inhomogeneous Heat Equation (25 points).

(a) Find a formal solution to

$$u_t = u_{xx} + xu, \quad -\infty < x < \infty, \quad t > 0,$$

subject to the arbitrary initial condition $u(x, 0) = f(x)$, by means of Fourier transforms. (Hint: You may find it useful to use the method of characteristics.)

(b) Under what conditions can the formal solution in part (a) be made rigorous. (Hint: consider the function $f(x)$, its derivative or its integral, or ....)