1) Let \( u(x, t) \) be defined by the formal integral,
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(m) e^{-m^2 t} e^{imx} \, dm
\]
where \( \rho(m) \) is bounded and continuous for all \( m \).

Comment: If you use any parts of the Riemann-Lebesgue lemma in your arguments below, you must prove them.

a) By formally differentiating under the integral sign, show that \( u(x, t) \) satisfies the heat equation, \( \partial_t u = \partial_x^2 u \) provided the derivatives exist.

b) Show that for any \( t > 0 \), \( \partial_t u \) is uniformly bounded in \( x \). [The argument for \( \partial_x^2 u \) is identical. This justifies the calculation in (a).]

c) Show that for \( t > 0 \), \( u(x, t) \to 0 \) as \( x \to \pm \infty \).

2) The electromagnetic field in a medium obeys the wave equation with a speed \( c \) that depends upon the medium. Suppose the medium has a jump discontinuity at \( x = 0 \), with speed \( c_1 \) for \( x < 0 \), and with speed \( c_2 \) for \( x > 0 \):
\[
E_{tt} - c_1^2 E_{xx} = 0, \quad x < 0, \quad t > 0 \\
E_{tt} - c_2^2 E_{xx} = 0, \quad x > 0, \quad t > 0
\]

The electric field and its spatial derivative are continuous across \( x = 0 \):
\[
E(0^-, t) = E(0^+, t) \\
E_x(0^-, t) = E_x(0^+, t)
\]

Suppose that the field is initially a localized pulse \( P(x) \) which is nonzero only in a domain \( 0 < x < b \), and that \( E_t \) is initially zero.

a) Find \( E(x,t) \). [Hint: you may find it useful to solve the problem in the two domains \( x \geq 0 \), and later impose the conditions at \( x = 0 \)]

b) When does the pulse first enter the region \( x < 0 \)?

c) What is the width of the pulse once it has entered the region \( x < 0 \)?
3) Let $u(x, t)$ be the temperature in a rod of length $L$. It satisfies the following problem.

\[
\begin{align*}
\partial_t u &= k \partial_x^2 u, & 0 < x < L, \ t > 0 \\
u &= 0, & x = 0, \ t > 0 \\
\partial_x u &= \tau & x = L, \ t > 0 \\
u(x, 0) &= 0, & 0 < x < L, \ t = 0,
\end{align*}
\]

where $\tau$ is a constant.

a) If $\tau > 0$, is heat entering the rod, leaving the rod or neither?
b) What is the steady-state temperature distribution in the rod?
c) What is the maximum temperature in the rod at steady state? Where does it occur?
d) Find $u(x, t)$ for $t > 0$.
e) Approximately how long is required before the temperature in the rod is within 1% of its steady-state value at every $x$? Justify your answer.

4) Let $z(x, y)$ solve the following first-order equation

\[
\begin{align*}
xz \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} &= xy & -\infty < y < \infty, \ x > 1 \\
z(1, y) &= \tanh(y) & x = 1.
\end{align*}
\]

a) Find $z(x, y)$ in the half-plane \{-\infty < y < \infty, \ x > 1\}.
b) Find the region (i.e. point(s), curve(s), area(s)) of this plane were $z = 0$.
c) Find the region where $z = 1$.
d) Do the characteristics ever cross for $x \geq 1$? If so, what is the point of crossing closest to the line $x = 1$.

5) Let $u(x, y)$ satisfy Laplace’s equation in a semi-infinite strip:

\[
\begin{align*}
\Delta u &= 0 & 0 < x < \infty, \ 0 < y < H \\
u(0, y) &= 1 & 0 < y < H \\
u(x,0) &= \begin{cases} 
1 & 0 < x < L \\
0 & x > L
\end{cases} \\
u(x,H) &= 0, & 0 < x < \infty \\
u(x,y) \text{ bounded as } x \to \infty & 0 < y < H
\end{align*}
\]

a) Obtain a formal solution to this equation.
b) Evaluate $u(L, H)$.
c) Evaluate $u_x(L, H)$
d) Does $u$ have a finite limit as $x \to \infty$? If so, what is it? Justify your answer.