Preliminary Examination in Numerical Analysis
Department of Applied Mathematics
Tuesday, January 13, 2009 (10 am - 1 pm)

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

1. Root Finding.
   (a) Describe Newton’s method for finding a root of $f : \mathbb{R} \to \mathbb{R}$.
   (b) State and prove a theorem about quadratic convergence of the method. Be as general as you can and include whatever assumptions you need on the derivatives of $f$ and the initial guess.

2. Numerical Quadrature.
   (a) Derive Simpson’s rule and its error formula for approximating $\int_{a}^{b} f(x) dx$ using the values of $f : \mathbb{R} \to \mathbb{R}$ at $a$, $(a + b)/2$, and $b$.
   (b) Derive the composite Simpson rule and its error formula for approximating $\int_{a}^{b} f(x) dx$ using the values of $f$ at $a + jh$, $j = 0, 1, ..., n$, where $h = (b - a)/n$.

3. Interpolation/Approximation.
   (a) Obtain the minimax first-degree polynomial approximation to $f(x) = \frac{1}{1+x}$ on $[0, 1]$.
   (b) Formulate the theorem describing properties of the minimax error.

4. Linear Algebra.
   (a) Define the concept of a vector norm on $\mathbb{R}^n$.
   (b) Does $\|x\| = \sup_{p \geq 1}(\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}}$ define a vector norm on $\mathbb{R}^n$? (You may use the fact that $\|x\|_p = (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}}$ is a norm.)
   (c) Does $\|x\| = \lim_{p \to \infty}(\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}}$ define a vector norm on $\mathbb{R}^n$?

5. Numerical ODE’s.
   Consider the two step method (Adams-Bashforth)
   
   $$y_{n+2} = y_{n+1} + h \left[ \frac{3}{2} f(t_{n+1}, y_{n+1}) - \frac{1}{2} f(t_n, y_n) \right].$$

   Show that it is convergent, find its order, and sketch its region of absolute stability. Make sure you state relevant theorems.

   Consider the heat equation
   
   $$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x} \left( a(x) \frac{\partial \phi}{\partial x} \right),$$

   with initial condition $\phi|_{t=0} = \phi_0$ and periodic boundary conditions on the interval $[0, 1]$. Fully describe the Crank- Nicolson scheme for this problem, using a staggered grid for the spatial operator. Taking $a(x) = 1$, show that the scheme is unconditionally stable.