Instructions:

You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems. Write your name on your exam. Each problem is worth 20 points.

1. Let $\Omega$ be the closed unit ball, $B_1(0)$, in $\mathbb{R}^n$ (with Euclidean norm). Let $T : \Omega \to \Omega$ satisfy $d(Tx, Ty) \leq d(x, y) \quad \forall x, y \in \Omega$.

Show that there exists at least one fixed point of $T$. (Hint: consider $T_k = (1 - \frac{1}{k}T)$.)

2. Let $F$ be a linear operator on a Hilbert space, $H$.

(a) Define

i. Point spectrum
ii. Continuous spectrum
iii. Residual spectrum

- Consider the operator $F : L^2[0, 1] \to L^2[0, 1]$ defined as $Fu = au$ where

$$a(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \leq x < \frac{3}{4} \\ 1 & \frac{3}{4} \leq x \leq 1 \end{cases}$$

(draw the graph of $a(x)$)

- What is the

i. Point spectrum
ii. Continuous spectrum
iii. Residual spectrum

Explain your answer.

3. (a) State the Hahn-Banach theorem.

(b) Let $Y$ be a linear subspace of a Banach space $X$, and let $x \in X$ be an element such that $d = \inf_{y \in Y} ||y - x|| > 0$.

Prove that there exists a functional $\varphi \in X^*$ such that

$$\varphi(x) = 1, ||\varphi|| = 1/d, \varphi(y) = 0, \quad \text{for all } y \in Y.$$ 

4. (a) State the Hölder inequality for Lebesgue measurable functions on $\mathbb{R}$.

(b) Let $p$ be a real number such that $1 < p < \infty$. Prove the triangle inequality in $L^p(\mathbb{R})$. (You may use the Hölder inequality.)

5. Evaluate the following limit by identifying it with a definite integral:

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k + \sqrt{n}k}$$
Solutions

Problem 1 Solution: For each $T_k$, we have

$$d(T_k x, T_k y) \leq (1 - \frac{1}{k})^2 d(x, y).$$

Thus, $T_k$ is a contraction. Let $x_k$ be the unique fixed point of $T_k$. Since $\Omega$ is compact, the sequence $(x_k) \subset \Omega$ has a convergent subsequence, $(x_{k_j}) \to x \in B_1(0)$.

Note that $\|T - T_k\| \to 0$. Given $\epsilon > 0$ choose $N > 0$ such that for $j > N$ we have

$$\|T - T_{k_j}\| \leq \epsilon/2$$

Thus,

$$\|T x - x\| = \|T x - T_{k_j} x - T_{k_j} x + T_{k_j} x_{k_j} - x\| \leq \|T x - T_{k_j} x\| + \|T_{k_j} x - T_{k_j} x_{k_j}\| + \|T_{k_j} x_{k_j} - x\| \leq \|T x - T_{k_j} x\| + 2\|x - x_{k_j}\| \leq \epsilon$$

Problem 2 Solution:

(a) i. Point spectrum: $\lambda$ is in the point spectrum if $(F - \lambda I)$ is not 1-to-1.

ii. Continuous spectrum: $\lambda$ is in the continuous spectrum if $(F - \lambda I)$ is 1-to-1 but not onto and the range of $(F - \lambda I)$ is dense in $H$.

iii. Residual spectrum: $\lambda$ is in the residual spectrum if $(F - \lambda I)$ is 1-to-1 but not onto and the range of $(F - \lambda I)$ is not dense in $H$.

(b) The spectrum of $F$ is:

i. Point spectrum: $\lambda \in \{1/2, 1\}$
   - For $\lambda = 1/2$, any function with support only in $(1/2, 3/4)$ is in the null space.
   - For $\lambda = 1.0$, any function with support only in $(3/4, 1.0]$ is in the null space.

ii. Continuous spectrum $\lambda \in [0, 1/2)$
   - For $\lambda \in [0, 1/2)$ the function $a(x) - \lambda = 0$ at $x = \lambda$. The function $v(x) = 1$ is not in the range of $F$ because $u = \frac{1}{a(x) - \lambda} \not\in L^2[0, 1]$. However, the sequence
     $$u_n = \begin{cases} 0 & x \in [\lambda - 1/n, \lambda + 1/n] \\ \frac{1}{a(x) - \lambda} & x \not\in [\lambda - 1/n, \lambda + 1/n] \end{cases}$$

   yields $F(u_n) \to v$. A similar construction proves that the range of $F$ is dense.

iii. Residual spectrum $\emptyset$
   - There are no values of $\lambda$ for which $F$ is 1-to-1 but the range is not dense.
Problem 3 Solution:
(a) See Theorem 5.58 in Hunter’s book.
(b) Consider the linear space \( \mathbb{Z} \) spanned by \( Y \) and \( x \). Since \( x \notin Y \), any element \( z = y + \lambda x \) with \( y \in Y \) and \( \lambda \in \mathbb{C} \). Consequently, we can define a functional \( \varphi \in \mathbb{Z}^* \) by
\[
\varphi(z) = \varphi(y + \lambda x) = \lambda.
\]
Next we determine \( ||\varphi|| \). Using that \( \text{dist}(Y, x) = d \) we find that for any element \( z = y + \lambda x \) with \( \lambda \neq 0 \) we have
\[
||y + \lambda x|| = |\lambda| \left| \frac{1}{\lambda} y + x \right| \geq |\lambda| d.
\]
Since \( |\lambda| = |\varphi(y + \lambda x)| \) it follows that \( ||\varphi|| \leq 1/d \). To prove that \( ||\varphi|| \geq 1/d \), simply pick a sequence \( (y_n) \) in \( Y \) such that \( ||y_n - x|| \to d \) and note that
\[
1 = |\varphi(y_n - x)| \leq ||\varphi|| \ |y_n - x|.
\]
It follows that \( ||\varphi|| \geq 1/||y_n - x|| \) and taking limits we obtain \( ||\varphi|| \geq 1/d \).
Finally invoke the Hahn-Banach theorem to prove that \( \varphi \) can be extended to all of \( X \) without changing its norm.

Problem 4 Solution:
(a) See Theorem 12.54 in Hunter’s book.
(b) Suppose that \( f, g \in L^p(\mathbb{R}) \) and set \( q = p/(p - 1) \). Then
\[
||f + g||_p^p = \int |f + g|^p \\
\leq \int |f + g|^{p-1}(|f| + |g|) \quad \text{[Holder]} \\
\leq \left( \int |f + g|^{(p-1)q} \right)^{1/q} \left( \int |f|^p \right)^{1/p} + \left( \int |f + g|^{(p-1)q} \right)^{1/q} \left( \int |g|^p \right)^{1/p} \\
= \left( \int |f + g|^p \right)^{(p-1)/p} \left( \int |f|^p \right)^{1/p} + \left( \int |f + g|^p \right)^{(p-1)/p} \left( \int |g|^p \right)^{1/p} \\
= ||f + p||_p^{p-1} ||f||_p + ||f + p||_p^{p-1} ||g||_p.
\]
Dividing by \( ||f + g||_p^{p-1} \) yields
\[
||f + g||_p \leq ||f||_p + ||g||_p.
\]

Problem 5 Solution:
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k^{1+\sqrt{mk}}} = \int_0^1 \frac{1}{x^{1+\sqrt{x}}} \, dx = \int_0^1 \frac{1}{\sqrt{x}(1+\sqrt{x})} \, dx = \\
\text{(using } u = \sqrt{x} \text{ as the new variable, it becomes)} \\
= \int_0^1 \frac{2}{1+u} \, du = 2 \ln(1 + u)|_0^1 = 2 \ln 2
\]