Problem 1: Recall that if $I$ is an interval in $\mathbb{R}$, then any function $f \in L^p(I)$ satisfies
\[
\int_I |f(x)|^p \, dx < \infty, \quad \text{when } p \in [1, \infty),
\]
\[
\operatorname*{esssup}_{x \in I} |f(x)| < \infty, \quad \text{when } p = \infty.
\]
Analogously, any sequence $x = (x_1, x_2, \ldots) \in l^p$ satisfies
\[
\sum_{n=1}^{\infty} |x_n|^p \, dx < \infty, \quad \text{when } p \in [1, \infty),
\]
\[
\sup_{n \in \mathbb{N}} |x_n| < \infty, \quad \text{when } p = \infty.
\]

Four pairs of function spaces are given below. For each pair $[A, B]$, answer two questions:

$A \subseteq B \ ? \quad B \subseteq A \ ?$

If the statement is true, then prove it. If not, then give a counter example.

(1) $[l^1, l^2]$
(2) $[l^1, l^\infty]$
(3) $[L^1(0, 1), L^2(0, 1)]$
(4) $[L^1(\mathbb{R}), L^2(\mathbb{R})]$ 

Problem 2: Let $A$ and $B$ be the linear operators on $l^2$ defined by
\[
A : (x_1, x_2, \ldots) \rightarrow (x_1, \frac{1}{2} x_2, \ldots, \frac{1}{2^n} x_n, \ldots),
\]
and
\[
B : (x_1, x_2, \ldots) \rightarrow (0, x_1, \frac{1}{2} x_2, \ldots, \frac{1}{n} x_n, \ldots).
\]

Prove that $A$ and $B$ are compact. Prove that $B$ does not have any eigenvalues. Determine the spectrum of $B$.

Problem 3: Let $S$ denote the collection of all simply connected subsets of $\mathbb{R}^2$ that have smooth boundaries. Define a function $\varphi$ that maps a set $\Omega \in S$ to $\mathbb{R}$ via the formula
\[
\varphi(\Omega) = \int_{\partial\Omega} \mathbf{n}(x) \cdot \mathbf{F}(x) \, ds(x),
\]
where, for $x = (x_1, x_2) \in \mathbb{R}^2$
\[
\mathbf{F}(x) = (4x - 2xy^2, 4x - x^2y),
\]
where $\partial\Omega$ is the boundary of $\Omega$, where for $x \in \partial\Omega$, $\mathbf{n}(x)$ is the outwards pointing unit normal to $\partial\Omega$ at $x$, and where $ds(x)$ is the element of arclength along $\partial\Omega$. Determine a subset $\tilde{\Omega} \in S$ for which
\[
\varphi(\tilde{\Omega}) = \sup_{\Omega \in S} \varphi(\Omega).
\]
Problem 4: Set $I = [0,1]$ and consider the Banach space $X = C(I)$ (equipped with the uniform norm, as usual). Define for $n = 1, 2, 3, \ldots$ functionals $T_n \in X^*$ via the formula

$$T_n(f) = f(1/n).$$

Prove that the set $\Omega = \{T_n\}_{n=1}^\infty$ is not compact in the norm topology on $X^*$.

Problem 5: Consider the equation

(1) \hspace{1cm} u(x) + u(x)^2 + \int_0^x (1 + \cos(x + u(y))) \, dy = 0.

Prove that equation (1) has a continuously differentiable solution $u$ in some open interval around the origin.