Problem 1: In the $xy$-plane, let $C_1$ denote the straight line from $(-1, 0)$, to $(1, 0)$, and let $C_2$ denote the upper half of the unit circle, starting at $(1, 0)$ and ending at $(-1, 0)$, as shown below:

Let $F$ be the vector field $F(x, y) = [2x(1 + x^2y)e^{x^2y} - y, x^4e^{x^2y} + \frac{1}{3}x^3 + xy^2]$.

(a) Compute $\int_{C_1} F(x, y) \cdot dr$ by parameterizing the integral.

(b) Use Green’s theorem, and your answer from (a), to evaluate $\int_{C_2} F(x, y) \cdot dr$.

Problem 2: Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of functions in $C(\mathbb{R})$ such that $f_n(0) = 0$, and for each $x \in \mathbb{R}$, $\sup_{y \neq x} \frac{|f_n(x) - f_n(y)|}{|x - y|} \leq 1 + (\sin x)^2$.

(a) Prove that for any finite $A$, the sequence $(f_n)_{n=1}^{\infty}$ has a subsequence $(f_{n_j})_{j=1}^{\infty}$ that converges in $C([-A, A])$.

(b) Does the sequence $(f_n)_{n=1}^{\infty}$ necessarily have a subsequence that converges in $C(\mathbb{R})$? (Please provide a proof or a counter-example.)

Problem 3: Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of real numbers that satisfy

$$\sum_{n=1}^{\infty} n^c \alpha_n^2 < \infty,$$

for some real number $c$ such that $c > 3$. For $N = 1, 2, \ldots$, set

$$f_N(x) = \sum_{n=1}^{N} \alpha_n \sin(nx).$$

Prove that the sequence $(f_N)_{N=1}^{\infty}$ converges in $C^1([-\pi, \pi])$. (Prove the statement directly, without invoking Sobolev embedding-type theorems.)
Problem 4: Let $c_0$ be the subspace of $l^\infty$ defined as the set of all elements $x = (x_n)_{n=1}^\infty \in l^\infty$ such that $\lim_{n \to \infty} x_n = 0$. Prove that the topological dual of $c_0$ is isometrically isomorphic to $l^1$.

Problem 5: Let $X$ denote a Hilbert space, and let $T$ denote a self-adjoint compact operator on $X$ such that $\|T\| \leq 1/2$. Define $A_0 = I + T$, and set, for $n \geq 1$,

$$A_n = \frac{1}{2}(A_{n-1} + A_{n-1}^{-1}).$$

Prove that the sequence $(A_n)_{n=1}^\infty$ converges in the strong sense.