APPLIED ANALYSIS PRELIMINARY EXAMINATION
Jan. 8, 1994

Instructions: You have three hours to complete this exam. Please start each problem on a new page.

1. Assume \( f : \mathbb{R}^n \to \mathbb{R}^m \) is \( C^1 \) and \( L \) is an \( m \times n \) matrix.
   Show: \( \lim_{\|h\| \to 0} \frac{\|f(x+h) - f(x) - Lh\|}{\|h\|} = 0 \) iff \( L = \left( \frac{\partial f_i}{\partial x_j}(x) \right) \).

Do four of the following five problems:

2. Let \( f(x), \{f_n(x)\}_{n=1,2,\ldots} \) be lebesgue measurable functions on \([0, \infty]\), \( f_n(x) \geq -\frac{\cos x^2}{1+x^2} \) and \( f_n(x) \to f(x) \) almost everywhere on \([0, \infty]\). Show that: \( \liminf_{n \to \infty} \int_0^\infty f_n(x) \, dx \geq \int_0^\infty f(x) \, dx. \)
   Give an example to show the inequality may be strict.

3. Prove the existence and uniqueness of a continuous solution \( u \) to the initial value problem \( u''(t) + \frac{2}{t} u'(t) = -t^2 e^{u(t)}, u(0) = 0 \) for \( 0 < t < \delta \), for some small number \( \delta \).
   Show that \( u'(0) \) exists and compute \( u'(0) \).

4. \( a = (a_1, a_2, \ldots, a_n, \ldots) \) is a sequence of real numbers. Suppose that for any given sequence of real numbers \( b = (b_1, b_2, \ldots, b_n, \ldots) \) with \( \sum_{n=1}^\infty b_n^2 < \infty \), we know that \( \sum_{n=1}^\infty a_n b_n \) converges. Show that \( \sum_{n=1}^\infty a_n b_n^2 < +\infty \).

5. \( K(x, y) = \sin x \sin y + \cos 2x \cos 2y \). Let \( X = L^2[0, 2\pi] \). \( T : X \to X \) is defined by \( (Tf)(x) = \int_0^{2\pi} K(x, y) f(y) \, dy \).
   a) Is \( T \) a compact operator? (Prove your conclusion).
   b) Is \( T \) a self-adjoint operator? (Prove your conclusion).

6. For \( T, X \) as defined in previous problem,
   a) Find all eigenfunctions corresponding to the eigenvalue \( \pi \). Find the spectrum of \( T \).
   b) For what \( g \in X \) is the equation \( T f - f = g \) solvable?
   c) For what \( g \in X \), is the equation \( T f - \pi f = g \) solvable?