Problem 1: Let $H$ be an infinite dimensional Hilbert space. Recall that we say that a sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{B}(H)$ converges \textit{weakly} to an operator $A \in \mathcal{B}(H)$ if
\[
\lim_{n \to \infty} \langle A_n x, y \rangle = \langle Ax, y \rangle, \quad \forall \ x, y \in H,
\]
that $(A_n)_{n=1}^{\infty} \subset \mathcal{B}(H)$ converges \textit{strongly} to an operator $A$ if
\[
\lim_{n \to \infty} \|A_n x - Ax\| = 0, \quad \forall \ x \in H,
\]
and that $(A_n)_{n=1}^{\infty}$ converges \textit{in norm} to $A$ if
\[
\lim_{n \to \infty} \|A_n - A\| = 0.
\]
(a) Define what it means for an operator in $\mathcal{B}(H)$ to be \textit{compact}.
(b) Suppose $(A_n)_{n=1}^{\infty}$ is a sequence of compact operators that converges to a limit $A$. For which of the three modes of convergence listed, if any, must $A$ be compact? If any mode of convergence is not sufficient for $A$ to be compact, provide a counterexample. Carefully motivate your statements.

Problem 2: Set $I = [-\pi, \pi]$, and suppose that $u : I \to \mathbb{C}$ is a continuously differentiable function. Suppose further that both $u$ and its derivative are periodic, so that $u(\pi) = u(-\pi)$ and $u'(\pi) = u'(-\pi)$, and that its second derivative exists almost everywhere and satisfies
\[
u''(x) = \begin{cases} 
1 & \text{when } |x| < \pi/2, \\
-1 & \text{when } |x| > \pi/2.
\end{cases}
\]
Do the conditions specified uniquely determine $u$? Do they uniquely determine for which $s$ the function $u$ belongs to the Sobolev space $H^s(I)$?

Problem 3: Two unrelated limits.
(a) Find the following limit:
\[
\lim_{k \to \infty} \int_{0}^{1} \frac{1 + kx^2}{(1 + x^2)^k} \, dx
\]
(b) Let $\{b_n\}$ be a bounded sequence of nonnegative numbers and let $r \in [0, 1)$. Define
\[
s_n = \sum_{k=1}^{n} b_k r^k \text{ for } n = 1, 2, 3, \ldots. \text{ Discuss the convergence/divergence of } \{s_n\}.
Problem 4: Hermite polynomials
(a) Define the set of Hermite polynomials, \( \{H_n(x)\} \), by
\[
H_n(x) := (-1)^n e^{x^2} \left( \frac{d}{dx} \right)^n e^{-x^2}, \quad \text{for } n = 0, 1, 2, 3, \ldots
\]
Find \( H_n(x) \) explicitly for \( n = 0, 1, 2, 3 \).
(b) Fact (no work required by you): \( H'_n(x) = 2nH_{n-1}(x) \). Now prove that these polynomials satisfy a weighted orthogonality relation on \( (-\infty, \infty) \):
\[
\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2} \, dx = 0 \quad \text{for } n \neq m
\]
Hint: This condition can be rewritten as
\[
(-1)^m \int_{-\infty}^{\infty} H_n(x) \left( \frac{d^m e^{-x^2}}{dx^m} \right) \, dx = 0
\]
For \( m > n \), integrate by parts \( n \) times and evaluate the resulting integrals.
(c) Define
\[
\varphi_n(x) := e^{-x^2/2}H_n(x) \quad \text{for } n = 0, 1, 2, \ldots
\]
Prove that the functions \( \{\varphi_n(x)\}_{n=0}^{\infty} \) are mutually orthogonal in \( L^2(-\infty, \infty) \).
(d) Fact (no work required by you): These orthogonal functions can be normalized. Therefore, assume that for a specific set of constants \( \{c_n\} \), \( \{c_n\varphi_n(x)\}_{n=0}^{\infty} \) form an orthonormal set in \( L^2(-\infty, \infty) \). Now let \( f(x) \) be an \( L^2 \) function. If
\[
f(x) = \sum_{n=0}^{\infty} f_n(e_n\varphi_n(x))
\]
find an explicit formula for the coefficient \( f_n \).

Problem 5: Let \( X \) be a normed linear space. A subset \( C \subset X \) is called convex if \( \forall x, y \in C \) and \( \forall 0 \leq \lambda \leq 1 \) then \( \lambda x + (1 - \lambda)y \in C \). A functional \( f \) on \( X \) is sub-additive if \( f(x + y) \leq f(x) + f(y) \forall x, y \in X \), and it is sub-linear if it is sub-additive and positive homogenous. A hyper-plane in \( \mathbb{R}^n \) defined by a normal vector \( a \in \mathbb{R}^n \) and offset \( \beta \in \mathbb{R} \) is the set \( \{x \in \mathbb{R}^n : \langle a, x \rangle = \beta \} \). A hyper-plane in \( X \) defined by a linear functional \( \phi \) and offset \( \beta \) is the set \( \{x \in X : \phi(x) = \beta \} \).
(a) The gauge function of a set \( C \) is defined \( \gamma_C(x) = \inf \{\lambda > 0 : x \in \lambda C\} \). Clearly \( \gamma_C \) is positive homogenous. Show that if \( 0 \in \text{int}C \) then \( \gamma_C \) is finite, and if \( C \) is convex then \( \gamma_C \) is sub-additive.

(b) A slightly stronger version of the Hahn-Banach theorem, in comparison to the version in Hunter and Nachtergaele’s book, is the following: let \( Y \) be a subspace of \( X \) and \( \psi \) a linear functional on \( Y \) such that it is dominated by a sub-linear functional \( p \), i.e., \( \psi(x) \leq p(x) \forall x \in Y \). Then \( \psi \) can be extended to a functional \( \Psi \) on \( X \) such that \( \Psi(x) \leq p(x) \forall x \in X \).
Show this implies the book’s version of the theorem: if \( \psi \) is a bounded linear functional on \( Y \), then it can be extended to a bounded linear functional \( \Psi \) on \( X \) such that \( \|\Psi\| = \|\psi\| \).

(c) Prove the hyperplane separation lemma (a variant of this is known as the geometric Hahn-Banach theorem): let \( C \subset X \) be a convex set with non-empty interior, and \( d \in X \setminus C \). Prove there exists a separating hyperplane between \( C \) and \( d \), i.e., there exists a linear function \( \psi \) on \( X \) such that \( \psi(c) \leq \psi(d) \forall c \in C \).
Hint: Define an appropriate one-dimensional subspace and use part (a) of the problem as well as the stronger version of the Hahn-Banach theorem.