**Instructions:** You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems. Write your name on your exam. Each problem is worth 20 points.

1. (a) Consider the vector field \( \mathbf{A} = (x + 1, y)e^{-x^2 - 2x - y^2} \). Prove that \( \nabla \times \mathbf{A} = 0 \). (4p)

   (b) Let \( G \) denote the set consisting of all \( C^1 \) curves in \( \mathbb{R}^2 \) of finite length, and define for \( \Gamma \in G \) the function \( f \) by letting
   \[
   f(\Gamma) = \int_{\Gamma} \mathbf{A} \cdot d\mathbf{x}.
   \]
   Determine \( \max_{\Gamma \in G} f(\Gamma) \), if it exists (if not, prove that it does not). (8p)

   (c) Determine \( \sup_{\Gamma \in G} f(\Gamma) \), if it exists (if not, prove that it does not). (8p)

2. Evaluate the limit
   \[
   \lim_{n \to \infty} n \int_{0}^{\infty} \frac{\sin(x/n)}{x(1 + x^2)} \, dx.
   \]
   Make sure to justify your calculation.

3. (a) Consider the Banach space \( X = l^3 \), and its subset \( S = \{ x \in X : ||x|| = 1 \} \). What is the weak closure of \( S \)? Prove your conclusion.

   (b) Fix a non-zero vector \( u \in X \) and define the operator \( T_n \) on \( X \) by setting \( T_n x = u x_n \), for \( x = (x_1, x_2, \ldots) \in X \). Prove that \( T_n \to 0 \) strongly in \( X \).

   (c) Define \( T_n^* \) (both its action and its range) and specify in what sense (if any) the sequence \( \{T_n^*\}_{n=1}^{\infty} \) converges to zero.

4. Consider the integration operator \( T \) that is defined by
   \[
   [T u](x) = \int_{0}^{x} u(s) \, ds.
   \]
   Prove that \( T \) is a compact operator on \( C([0, 1]) \) and determine its spectrum.

5. Use the contraction mapping theorem to prove the existence of a \( C^2 \) function \( \phi(x, y) \) solving the equation
   \[
   \cos \phi(x, y) + (x^2 + y^2 + 4)\phi(x, y) + \phi^2(x, y) + 3x^3 - 1 = 0 \]
   on the closed ball of radius \( \delta \) and centered at \((0, 0)\) for some \( \delta > 0 \) with \( \phi(0, 0) = 0 \).