APPLIED ANALYSIS PRELIMINARY EXAMINATION
August 19, 1994

Instructions: You have three hours to complete this exam. Please start each problem on a new page.

Do one of the following two:

1. Two functions $f$ and $g$ are defined as follows:
   \[ f : \mathbb{R}^2 \to \mathbb{R}^3, \quad f(x) = (x_1 \cos(x_2), \ x_1^2 + x_2^2, \ x_1 x_2) \]
   \[ g : \mathbb{R}^3 \to \mathbb{R}^2, \quad g(x) = (x_1 x_2 \cos(x_3), \ x_1 x_2 x_3) \]

   Compute the Jacobian matrix $D(gof)(0, \pi)$ using the chain rule. (No credit for forming the composition and then computing the Jacobian.)

2. Given $f, g : \mathbb{R}^3 \to \mathbb{R}^1$ continuously differentiable functions

   Let $X = \{ x \in \mathbb{R}^3 : g(x) = 0 \}$.

   Suppose that $x_0$ is a relative minimum of the function $f$ restricted to $X$. Show that $\nabla f(x_0)$ is a scalar multiple of $\nabla g(x_0)$, provided $\nabla g(x_0) \neq 0$.

Do four of the following five:

3. Suppose $f_n, f \in L^2(1, +\infty)$ and $f_n \to f$ pointwise almost everywhere. Find the relationship between the following integrals and prove your claims.

   \[ \int_1^{+\infty} \frac{f^2}{1 + f^2}, \quad \lim_{n \to \infty} \int_1^{+\infty} \frac{f_n^2}{1 + f_n^2}, \quad \lim_{n \to \infty} \int_1^{+\infty} \frac{f_n^2}{1 + f_n^2} \]

   \[ \int_1^{+\infty} \frac{f^2}{1 + x^2 f^2}, \quad \lim_{n \to \infty} \int_1^{+\infty} \frac{f_n^2}{1 + x^2 f_n^2}, \quad \lim_{n \to \infty} \int_1^{+\infty} \frac{f_n^2}{1 + x^2 f_n^2} \]
4. Let \( H = L^2[0, 2\pi] \). An operator \( A : H \to H \) is defined by

\[
(Af)(x) = \int_0^{2\pi} (\sin x \cos y + \cos y)f(y) \, dy.
\]

Answer the following questions with proofs:

(a) Is \( A \) self-adjoint?

(b) Is \( A \) compact?

5. Given functions \( f_n, f \in H = L^2[0, 1] \), \( f_n \rightharpoonup f \)

Show that \( f_n \rightharpoonup f \) if and only if

\[
\lim_{n \to \infty} \int |f_n|^2 = \int f^2.
\]

6. Prove there exists a unique continuous solution \( u \) to the ODE:

\[
u'(t) = \frac{\sin(t \cdot u(t))}{2t} \quad \text{for} \quad 0 < t \leq 1\]

which satisfies the initial condition \( u(0) = 1 \).

7. Let \( S \) be the unit sphere in \( L^2[0, 2\pi] \).

Define a function in \( G : S \to R^1 \) by \( G(f) = \int_0^{2\pi} f(x) \sin(x) \, dx \).

Calculate the sup \( \{G(f) : f \in S\} \) and calculate the inf \( \{G(f) : f \in S\} \). Justify your calculations!