INSTRUCTIONS: Books, notes, and electronic devices are not permitted. This exam is worth 100 points. Box your final answers. Write neatly, top to bottom, left to right, one problem per page. A correct answer with incorrect or no supporting work may receive no credit. **SHOW ALL WORK**

1. (24 points; 6, 6, 6, 6) Derivate the following functions with respect to $x$:

   (a) $y = x\sqrt{x}$  
   (b) $y = \frac{1 - \cos x}{x}$  
   (c) $y = (x + x^{-1})^2$  
   (d) $y = \sqrt{3x^2 - 5x - 8}$

Solution:

(a) $y = x\sqrt{x} = x^{\frac{3}{2}} \implies y' = \frac{3}{2} x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2}$

(b) $y' = \frac{(x \sin x) - (1 - \cos x)}{x^2} = \frac{x \sin x + \cos x - 1}{x^2}$

(c) $y' = 2(x + x^{-1})(1 - x^{-2}) = 2 \left( x + \frac{1}{x} \right) \left( 1 - \frac{1}{x^2} \right)$

(d) $y' = \frac{1}{2} (3x^2 - 5x)^{-\frac{1}{2}} (6x - 5) - 8$
2. (18 points;6,6,6) Consider the following functions:

\[ f(x) = \frac{|x+1|}{x} \quad \text{and} \quad g(x) = \sqrt{x^2 - 1} \quad \text{and} \quad h(x) = \begin{cases} x^2, & x \leq -1 \\ -x^2 + 1, & -1 < x < 1 \\ -1 + x^2, & x > 1 \end{cases} \]

(a) Find \( f'(-2) \).
(b) Find \( g'(1) \).
(c) Find \( h'(-1) \).

Solution:

(a) For \( x = -2 \) we have \( f(x) = \frac{-(x+1)}{x} = -1 - \frac{1}{x} \) so \( f'(x) = \frac{1}{x^2} \) and \( f'(-2) = \frac{1}{4} \).

(b) Since \( g(x) \) is undefined for \( x < 1 \), we know \( g'(1) \) does not exist.
\[ g'(x) = \frac{1}{2} (x^2 - 1)^{-\frac{1}{2}} (2x) = \frac{x}{\sqrt{x^2 - 1}} \implies g'(1) = \emptyset \]

(c) \( h(x) \) is two different functions to the left and the right of \( x \).
\[ h'(x) = \begin{cases} 2x, & x < -1 \\ -2x, & -1 < x < 1 \\ 2x, & x > 1 \end{cases} \]

Note that \( \lim_{x \to -1^-} h'(x) = \lim_{x \to -1^-} (2x) = -2 \) while \( \lim_{x \to -1^+} h'(x) = \lim_{x \to -1^+} (-2x) = 2 \), therefore \( h'(x) = \emptyset \).
3. (18 points:6,6,6) Consider the following graph of \( f(x) = 2x^3 - 3x^2 + 3 \) with secant line \( PQ \):

(a) Find the average rate of change of \( f(x) \) between points \( P \) and \( Q \) when \( a = h = 1 \).
(b) Find the instantaneous rate of change of \( f(x) \) at point \( P \) when \( a = 1 \) and \( h = 0 \).
(c) What is the rate of change of \( f(x) \) at \( x = a \)?

Solution:

(a) \( P(1, 2) \) and \( Q(2, 7) \) means \( \frac{\Delta y}{\Delta x} = \frac{7-2}{2-1} = 5 \)

(b) \( f'(x) = 6x^2 - 6x \implies f'(1) = 6 - 6 = 0 \)

(c) \( f'(a) = 6a^2 - 6a = 6a(a-1) \)
4. (7 points) Find constants $A$, $B$, and $C$ such that the function $y = Ax^2 + Bx + C$ satisfies the differential equation $y'' + y' - 2y = (x + 1)(x - 8)$.

Solution: $y = Ax^2 + Bx + C \implies y' = 2Ax + b \implies y'' = 2A$

$y'' + y' - 2y = (x + 1)(x - 8) \implies 2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2 - 7x - 8$

$(-2A)x^2 + (2A - 2B)x + (2A + B - 2C) = (1)x^2 + (-7)x + (-8)$

$-2A = 1$ and $2A - 2B = -7$ and $2A + B - 2C = -8$

$A = -\frac{1}{2}$ and $B = 3$ and $C = 5$
5. (8 points) Balloon Advertising Company sells a giant spherical balloon that can be inflated to vary from a 2 foot diameter to a 6 foot diameter. Gas is pumped into the balloon at \( \frac{4\pi}{3} \) ft\(^3\)/sec. How fast is the radius growing at the moment the volume is \( \frac{32\pi}{3} \) ft\(^3\)?

**Solution:**

\[
V = \frac{4}{3}\pi r^3 \implies \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 3 \text{ therefore, } \frac{dr}{dt} = \frac{3}{4\pi r^2}
\]

\[
V = \frac{4}{3}\pi r^3 = \frac{32\pi}{3} \implies r^3 = \frac{32\pi^3}{3\pi^4} \implies r^3 = \frac{32}{4} = 8 \implies r = 2
\]

\[
\frac{dr}{dt} = \frac{3}{4\pi(2)^2} = \frac{3}{16\pi}
\]
6. (7 points) Consider the following solid. All corners are $90^\circ$ angles. Suppose $x$ is continuously increasing. How fast is the volume of the object changing with respect to $x$ when the volume is 21?

![Diagram of a solid with dimensions $2x - 1$, $x + 1$, and $x + 3$.]

**Solution:** $V = 3x(2x - 1) + x^2(x + 1) = 6x^2 - 3x^3 + x^2 = x^3 + 7x^2 - 3x \implies \frac{dV}{dx} = V'(x) = 3x^2 + 14x - 3$

When Volume equals 21 we have $x^3 + 7x^2 - 3x = 21$ or $x^3 + 7x^2 - 3x - 21 = 0 \implies x^2(x + 7) - 3(x + 7) = 0 \implies (x^2 - 3)(x + 7) = 0 \implies x = \sqrt{3}$

We only consider $x = \sqrt{3}$ since $x = -7$ and $x = -\sqrt{3}$ would create sides of negative length.

Now, $V(\sqrt{3}) = 3(\sqrt{3})^2 + 14(\sqrt{3}) - 3 = \frac{6 + 14\sqrt{3}}{units^3}$.
7. (18 points; 6, 6, 6) Answer the following:
(a) Consider the relationship $3x^2 + 3y^2 + 6x - 12y = 0$. What is $\frac{dy}{dx}$?

(b) Find $\frac{dy}{d\theta}$ if $y = (\tan \theta \sin \theta + \csc \theta)$.

(c) Given that $T = \frac{A-P}{Pr}$, suppose $T$ and $A$ are constants, while $r$ and $P$ are variable. Describe how fast $P$ is changing with respect to $r$.

Solution:

(a) Implicit differentiation produces: $6x + 6yy' + 6 - 12y' = 0$

$y'(6y - 12) = -6x - 6$

$y' = \frac{-6(x + 1)}{6(y - 2)} = \frac{-1}{y - 2} \frac{x + 1}{2 - y}$

(b) $y' = \tan \theta \cos \theta + \sin \theta \sec^2 \theta - \csc \theta \cot \theta$ or perhaps, $\sin \theta + \tan \theta \sec \theta - \csc \theta \cot \theta$

(c) $T = \frac{A-P}{Pr}$

$TPr = A - P$

$TPr + P = A$

$P(Tr + 1) = A$

$P(r) = \frac{A}{Tr+1}$

$P'(r) = \frac{(Tr + 1) \ast 0 - At}{(Tr + 1)^2} = \frac{-At}{(Tr + 1)^2}$

Alternatively, $P(r) = A(Tr + 1)^{-1} \implies P'(r) = -A(Tr + 1)^{-2} = \frac{-At}{(Tr + 1)^2}$

END of Exam