Exam #2
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INSTRUCTIONS: On the front of your bluebook please print your name, student ID, date and course code. Show all your work in your bluebook. Please start each new problem on a new page. Solve the problems in the same order as they are requested. A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. Textbooks, class notes, graphing or programmable calculators, and crib sheets are not permitted.

1. (40 points) Let $(N(t))_{t \geq 0}$ denote a homogeneous Poisson (point) process with intensity $\lambda > 0$, and let $T_1, T_2, \ldots$ denote the arrival times of the process. Based on this, answer the following questions. Do not justify your answers!

(a) What’s the distribution of $T_3$?
(b) What’s the probability that $(T_4 - T_3) > 2$?
(c) What’s the probability that $N(5) - N(3) = 2$?
(d) What’s the probability of $N(5) - N(3) = 2$ given that $N(2) = N(1)$?
(e) What’s the probability of $N(5) - N(3) = 2$ given that $N(6) = 4$?
(f) What’s the expected value of $N(5) - N(3)$ given that $N(4) = 6$?
(g) What’s the expected value of $T_3$ given that $N(2) = 0$?
(h) What’s the expected value of $T_3$ given that $N(2) = 3$?

2. (30 points) Consider the state space $S = \{1, 2\}$ and the probability transition matrix:

$$p := \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix}.$$ 

Now consider the time-continuous Markov process $X = (X_t)_{t \geq 0}$ defined as follows: (i) when at a state 1, the chain stays for an exponential amount of time with mean $1/6$ before jumping (possibly immediate back to 1) according to the first-row of $p$; and (ii) when at state 2, the chain stays for an exponential amount of time with mean $1/8$ before jumping (possibly immediate back to 2) according to the second-row of $p$. Based on this description, respond:

(a) Explain why the rate-matrix of $X$ is

$$Q := \begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}.$$ 

(b) For a given $t > 0$, what’s the probability that $X_t = 1$ given that $X_0 = 1$?

(c) Determine the stationary distribution $\pi$ of the process.

(ONE MORE PROBLEM ON THE BACK!)
3. (30 points) Consider the Markov chain \( X = (X_n)_{n \geq 0} \) with state space \( S = \{0, 1, 2, 3, 4, 5, 6\} \) and probability transition matrix represented by the following (weighted directed) graph:

![Markov Chain Diagram]

Based on the above, respond:

(a) Explain why this chain has a unique stationary distribution \( \pi \) and determine it.
   **Hint.** Avoid the brute force approach!

(b) Does the limit \( \lim_{n \to \infty} P(X_{n+2} = 3 \mid X_n = 1, X_{n-1} = 0) \) exist? If so determine it. Justify!

(c) Does the limit \( \lim_{n \to \infty} P(X_{n+1} = 1, X_n = 0 \mid X_1 = 4, X_0 = 2) \) exist? If so determine it. Justify!

**DURATION: 90 MINUTES**
P1
(a) \( \text{Gamma}(3, 2) \)
(b) \( P(\text{Exponential}(2) > 2) = e^{-2a} \)
(c) \( P(\text{Poisson}(2a) = 2) = (20)^2 e^{-20}/2! \)
(d) \( P(N(15) - N(13) = 2) = \text{same as (c)} \)
(e) \( P(\text{Binomial}(4, \frac{1}{3}) = 2) = \left(\frac{4}{3}\right)^2 \left(\frac{2}{3}\right)^2 \)
(f) \( E(N(15) - N(13) | N(4) = 6) = E(N(15) - N(14)) + E(N(4) - N(3) | N(4) = 6) = E(\text{Poisson}(10)) + E(\text{Binomial}(6, \frac{1}{4})) \)
\[ = 71 + 6/4 = 71 + 1.5 \]
(g) \( E(T_3 | N(2) = 0) = 2 + E(T_3) = 2 + 3/2 \)
(h) \( E(T_3 | N(2) = 3) = E(U_{3,2}) \) where \( U_1, U_2, U_3 \) are iid \( \text{Unit}(10, 2) \) \nNote \( P(U_3 < x) = \left(\frac{x}{2}\right)^3 \), so \( P(U_{3,2} < x) = 3 \left(\frac{x}{2}\right)^2 \cdot \frac{1}{2} \) for \( 0 \leq x \leq 2 \). So
\[ E(U_{3,2}) = \int_0^2 \frac{3}{2} \left(\frac{x}{2}\right)^2 \cdot x \, dx = \frac{3}{8} \int_0^2 x^3 \, dx = \frac{3}{8} \left[ \frac{x^4}{4} \right]_0^2 = \frac{3}{2} \]

P2
(a) When at 0, the times of jump follow a HPP(6) but only 2/3 of these jumps move the process into 2. Since the thinned process has rate \( 6 \cdot \frac{2}{3} = 4 \), the rate of 0 is \( a_1 = 4 \).
Similarly, the rate out of 2 is \( b_2 = 3 \cdot \frac{2}{3} = 2 \).
So \( Q = \begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix} \)
(b) Using the forward equation:
\[ P_t'(1,1) = (P_t \cdot Q)_{1,1} = P_t(1,1) \cdot (-4) + P_t(1,2) \cdot 6 \]
But \( P_t(1,2) = 1 - P_t(1,1) \). So:
\[ P_t'(1,1) = 6 - 10 P_t(1,1) \]
\[ \text{i.e. } \frac{dP(t)}{dt} + 10P(t) = 0 \]

\[ \text{integrate both sides: } e^{10t} P(t) = 6 e^{10t} \]

\[ e^{10t} P(t) - P(0) = \int_0^t 6 e^{10s} ds. \]

Since \( P(0) = 1 \), we finally obtain that

\[ P(t) = e^{-10t} + 6 \int_0^t e^{10s} ds = e^{-10t} + 6 \left( \frac{e^{10s}}{10} \right) \bigg|_0^t \]

\[ \text{i.e. } P(t) = e^{-10t} + \frac{3}{5} (1 - e^{-10t}) = \frac{3}{5} + \frac{2}{5} e^{-10t} \]

E. Since \( \lim_{t \to \infty} P(t) = \frac{3}{5} \), we guess \( \lambda = \left( \frac{1}{3}, \frac{2}{3} \right) \).

Since

\[ \lambda \cdot Q = \left[ \frac{2}{3} \cdot 1 - \frac{3}{5} \cdot 2, \frac{3}{5} \cdot 1 - \frac{2}{3} \cdot 2 \right] = (0,0), \]

the guess is correct.

\[ P3 \]

\[ \text{We use detailed-balance condition using } \Theta \text{ to solve the recursion.} \]

\[ \xi(1) = \frac{1}{2} \]

\[ \xi(2) = \frac{1}{2} \]

\[ \xi(3) = \frac{1}{2} \]

\[ \xi(4) = \frac{1}{2} \]

\[ \xi(5) = \frac{1}{2} \]

\[ \xi(6) = \frac{1}{2} \]

\[ \text{and finally: } \xi(6) = \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \]
Since \( \frac{6}{10} \nu(0) = \nu(1) + 2 \nu(0) + \nu(0) + \frac{2}{3} \nu(0) = \frac{14}{3} \nu(0) \),
we need to select \( \nu(0) = \frac{3}{14} \).
Thus:
\[
\nu = \begin{bmatrix}
\frac{3}{14} & \frac{1}{7} & \frac{1}{7} & \frac{3}{28} & \frac{3}{28} & \frac{1}{7}
\end{bmatrix}
\]

\( \text{b) Yes, b/c it doesn't depend on } m \). Indeed:
\[
P(X_{n+2} = 3 \mid X_n = 1, X_{n-1} = 0) = P(X_{3} = 3 \mid X_0 = 1) = \frac{7}{2} \cdot \frac{3}{3} = \frac{1}{6}
\]
i.e.
\[
\lim_{m \to \infty} P(X_{n+2} = 3 \mid X_n = 1, X_{n-1} = 0) = \frac{1}{6}
\]

\( \text{c) Yes, b/c the chain is irreducible, periodic (b/c 0 has period 1) and has a stationary distribution. As a result:} \)
\[
\lim_{m \to \infty} P(X_{n+1} = 1, X_m = 0 \mid X_1 = 4, X_0 = 2)
\]
\[
= \lim_{m \to \infty} P(0, 1) \cdot \pi^{-1}(4, 0) = P(0, 1) \cdot \pi(0) = \frac{7}{2} \cdot \frac{3}{14} = \frac{1}{4}
\]