1. (20 points) Just provide answers, you don’t have to show any work and no partial credit will be given. Consider the system of equations

\[
\begin{align*}
    x_1 + x_2 + x_3 - x_4 &= 0 \\
    3x_1 + 3x_2 + x_3 - 5x_4 &= 0 \\
    4x_1 + 4x_2 + 2x_3 - 6x_4 &= 0
\end{align*}
\]

Let \( A \) be the coefficient matrix of the system. The RREF of \( A \) is

\[
\begin{pmatrix}
1 & 1 & 0 & -2 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

You do not have to verify this, just use it to answer the following questions:

(a) What is rank(\( A \))?

(b) What is a basis for Col(\( A \))? 

(c) Which vector space is Col(\( A \)) a subspace of?

(d) What is the dimension of the solution space of the given homogeneous system?

(e) What is a basis for the solution space of the given homogeneous system?

(f) Which vector space is the solutions space of the given homogeneous system a subspace of?

2. (28 points) Show your work in this problem. Consider the differential equation

\[
(1 + t^2)y'' - 2ty' + 2y = (1 + t^2)^2
\]

(a) Verify that the functions \( y_1(t) = t \) and \( y_2(t) = t^2 - 1 \) are solutions of the homogeneous version of the given differential equation.

(b) Use the Wronskian to verify that \( y_1 \) and \( y_2 \) are linearly independent.

(c) Use variation of parameters to find a particular solution \( y_p \) of the given differential equation. Simplify your answer for \( y_p \) as far as possible! You may use appropriate formulas but you must show your steps clearly.

(d) Write down the general solution of the given differential equation.
3. (24 points) Provide an answer with a short (correct!) justification for each question:

(a) Is the polynomial $3t^2 + 2t + 1$ in $\text{Span}\{t^2, t^2 - t - 1, t + 1\}$?

(b) Does the set of polynomials given in part (a), $\{t^2, t^2 - t - 1, t + 1\}$, form a basis of $\mathbb{P}_2$? Why or why not?

(c) **Use the definition of linear independence only** to determine if the vectors

\[
\begin{bmatrix}
1 \\
1 \\
4
\end{bmatrix}, \begin{bmatrix}
-3 \\
4 \\
2
\end{bmatrix}, \begin{bmatrix}
7 \\
-1 \\
3
\end{bmatrix}
\]

are linearly independent or not.

(d) Is the set $V$ of all $3 \times 3$ invertible matrices with real entries a vector space? Why or why not?

(e) Is the set $W = \{(x, y) \in \mathbb{R}^2 \mid x + y \geq 0\}$ a subspace of $\mathbb{R}^2$? Why or why not?

(f) Is the set $D$ of all $3 \times 3$ diagonal matrices with real entries a subspace of $M_{33}$? Why or why not?

4. (28 points) **For parts (a), (b), (c), do two things:** find the general solution $y_h$ of the homogeneous version of the given differential equation, and provide the correct type of “educated guess” you would use to find $y_p$, a particular solution of the given differential equation, with the method of undetermined coefficients. **You do not have to actually determine the coefficients in (a), (b), (c), just provide the right type of function.** You don’t have to provide any work and no partial credit will be given for parts (a), (b), (c).

(a) $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = (1 + t)e^{3t}$

(b) $y'' + 9y = 3t^2 \sin(3t)$

(c) $y'' + y' + y = 2t^2 + 2t + 2$

Now for part (d), solve the following IVP using the method of undetermined coefficients. Show all of your work clearly.

(d) $y'' - 2y' + y = 25 \cos(2t), \quad y(0) = 0, \quad y'(0) = 0.$