ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section, (4) your instructors’ name, and (5) a grading table. You must work all of the problems on the exam. Unless indicated, show ALL of your work in your bluebook and box in your final answer. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, calculators and ANY electronic devices are NOT permitted. A 8′′ × 11′′, two-sided, sheet of notes is allowed.

1. (a) (10 points) Use separation of variables to find the general solution to the differential equation

\[ t^2 y' - y' = 2te^{-y}. \]

(b) (5 points) Is the differential equation linear or nonlinear?

(c) (5 points) If the initial condition is \( y(t_0) = y_0 \), for what values of \( t_0 \) and \( y_0 \) is the initial value problem guaranteed to have a unique solution?

SOLUTION:

(a) The equation is separable:

\[ y' e^y = \frac{2t}{t^2 - 1}, \]

\[ \int e^y dy = \int \frac{2tdt}{t^2 - 1}, \]

\[ e^y = \ln|t^2 - 1| + C, \]

\[ y(t) = \ln(\ln|t^2 - 1| + C). \]

(b) The equation is nonlinear because of the \( y'e^y \) term.

(c) The equation can be written as \( y' = f(y, t) \), with \( f(y, t) = e^{-y}2t/(t^2 - 1) \). The function \( f \) is a continuous function of \( t \) and \( y \) except at \( t = \pm 1 \). Similarly, \( \partial f/\partial y = -f \) is continuous except at \( t = \pm 1 \). Therefore, by Picard’s theorem, the initial value problem is guaranteed to have a unique solution for all \( t_0, y_0 \) except when \( t_0 = \pm 1 \).

2. (30 points) Consider the following second order ODE

\[ y'' + 8y' + 16y = e^{-4t} + 32t. \]

(a) (10 Points) Solve for the homogeneous solution.

(b) (15 Points) Solve for a particular solution.

(c) (5 Points) Find the general solution.

Solution.

(a) \[ y_h(t) = c_1 e^{-4t} + c_2 te^{-4t} \]

(b) The method of undetermined coefficients gives

\[ y_p(t) = At^2 e^{-4t} + Bt + C. \]

This leads to

\[ (2A - 1)e^{-4t} + (16B - 32)t + (8B + 16C) = 0 \]

Thus \( A = 1/2, B = 2, C = -1 \), i.e.,

\[ y_p(t) = \frac{1}{2} t^2 e^{-4t} + 2t - 1 \]

(c) The general solution is

\[ y_g(t) = y_h + y_p = (c_1 e^{-4t} + c_2 te^{-4t}) + \left( \frac{1}{2} t^2 e^{-4t} + 2t - 1 \right) \]
3. (30 points)

(a) (8 points) Consider the equation \( y'' + 2y + 1 = 0 \). For each of the following terms, write True if the term describes the system, and False if it does not. 1 point each. No partial credit is awarded in this question.

1. Underdamped
2. Critically damped
3. Overdamped
4. Resonance
5. Beats
6. Solution decays to 0 as \( t \to \infty \)
7. Solution oscillates
8. Solution increases without bound

(b) (6 points) Consider the matrix \( A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \). Find the eigenvalues of \( A \).

(c) (16 points) Find the general solution to the equation \( \frac{d}{dt} \vec{x} = A \vec{x} \), where the matrix \( A \) is from part (b).

Solution:

(a) (8 pts) The equation as written is equivalent to \( y'' + 0y + 2y = -1 \) which, by observation (or method-of-undetermined-coefficients) has \( y_p = -1/2 \) as a particular solution, and the homogeneous equation has roots \( r = \pm \sqrt{2}i \), so it is oscillatory. It is ambiguous if this equation is underdamped (it is clearly undamped), so we accepted either True or False for 1. True terms are boxed:

1. Underdamped
2. Critically damped
3. Overdamped
4. Resonance
5. Beats
6. Solution decays to 0 as \( t \to \infty \)
7. Solution oscillates
8. Solution increases without bound

However, the equation was easy to interpret as \( y'' + 2y' + y = 0 \), which we treated with full credit. In this case, \( r = -1 \) is the only root of the characteristic equation, so the solution is \( y(t) = c_1 e^{-t} + c_2 te^{-t} \), which is critical damping. True terms are boxed:

1. Underdamped
2. Critically damped
3. Overdamped
4. Resonance
5. Beats
6. Solution decays to 0 as \( t \to \infty \)
7. Solution oscillates
8. Solution increases without bound

(b) (6 pts) This corresponds to the 2nd order equation from part (a) if you interpreted (a) as \( y'' + 2y' + y = 0 \), and has the same characteristic equation, so we have \( \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0 \), hence \( \lambda = -1 \) is the only root, with algebraic multiplicity 2.

(c) (16 pts) First, find eigenvectors. Choosing \( \lambda = -1 \), then \( A - \lambda I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \). You can eliminate a row right away, and it is then in RREF. The solution is thus any multiple of \( \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \), e.g., \( \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

We do not have two linearly independent eigenvectors, so we need to introduce another linearly independent solution. The form of the solution should be

\[ \vec{x}(t) = c_1 e^{-t} \vec{v} + c_2 e^{-t} (t \vec{v} + \vec{u}) \]

\[ (A - \lambda I) \vec{u} = \vec{v}. \]

We call \( \vec{u} \) a generalized eigenvector, and note that it is the solution to an inhomogenous equation, so it can be written in the form of \( \vec{u} = \vec{u}_p + c \cdot \vec{u}_h \) where \( \vec{u}_h = \vec{v} \) is the homogenous solution. Thus there is not a unique choice for \( \vec{u} \) as we can shift off any multiple of \( \vec{v} \).

Solving for \( \vec{u} \) should lead to a system like the following \( \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \) or \( \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \), depending on our choice of \( \vec{v} \). Using this first system, and observing the 2nd equation is redundant, we have \( 1 \cdot u_1 + 1 \cdot u_2 = 1 \), and letting \( u_2 = s \) we have \( \vec{u} = \begin{bmatrix} 1 - s \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + s \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + s \cdot \vec{v} \), so some common, valid
choices for \( \vec{u} \) (if you chose \( \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \)) are \( \vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \text{ or } \begin{bmatrix} 2 \\ -1 \end{bmatrix} \),

(if you chose \( \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \)) \( \vec{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \text{ or } \begin{bmatrix} -2 \\ 1 \end{bmatrix} \).

4. (30 points) For the questions in this problem, you do not need to show your work, and no partial credit will be given. If you do submit work, then [box] your answer, and know that your work will not be graded.

(a) (6 points) For what values of \( k \) are the following vectors a basis for \( \mathbb{R}^3 \)?

\[
\begin{bmatrix} 1 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ k \\ 0 \end{bmatrix}.
\]

(b) (6 points) What is the dimension of the vector space \( U = \{4 \times 4 \text{ upper triangular matrices}\} \)?

(c) (6 points) For the initial value problem \( y' = t + y, \ y(1) = 2 \), find the approximation to \( y(3) \) using Euler’s method with a time step \( h = 1 \).

(d) (6 points) Which of the following are vector subspaces of \( \mathbb{M}_{3 \times 3} \), the vector space of \( 3 \times 3 \) matrices with usual matrix operations? (i) \( 3 \times 3 \) matrices with at least a zero entry; (ii) \( 3 \times 3 \) matrices with integer entries; (iii) \( 3 \times 3 \) matrices with top left entry equal zero, \( a_{11} = 0 \).

(e) (6 points) True or False? If \( A \) and \( B \) are \( n \times n \) invertible matrices, then \( AB \) is invertible.

Solution:

(a) The determinant of the matrix \( A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & k \\ k & 1 & 0 \end{bmatrix} \) is \( |A| = 1(-k) + k(k - 12) = k^2 - 13k \). The three vectors are a basis when \( |A| \neq 0 \), or \( k \neq 0, k \neq 13 \).

(b) 10. Count the number of independent entries in a \( 4 \times 4 \) upper triangular matrix.

(c) We have \( y_{n+1} = y_n + h(t_n + y_n) \). Given \( y_0 = 2 \) and \( h = 1 \), we get \( y_1 = 2 + 1(1 + 2) = 5 \), \( y(3) \approx y_2 = 5 + 1(2 + 5) = 12 \).

(d) (i) No, not closed under addition. (ii) No, not closed under scalar multiplication. (iii) Yes.

(e) True, if \( |A| \neq 0 \) and \( |B| \neq 0 \), then \( |AB| = |A||B| \neq 0 \).
5. (30 points) Consider the system of differential equations
\[
\frac{dx}{dt} = 1 - x^2 - y^2, \\
\frac{dy}{dt} = 1 - x - y.
\]

(a) (10 Points) Determine equations for the horizontal and vertical nullclines of the given ODE system and find all the equilibrium points of the given ODE system.

(b) On one set of axes in phase space:
(i) (5 Points) Plot the nullclines and equilibrium points you found in part (a).
(ii) (10 Points) Put arrows on the nullclines indicating the direction of the slopes.
(e) (5 Points) Classify each equilibrium point as stable or unstable depending on the behavior of solutions that start nearby.

SOLUTION:
(a) To determine the v-nullclines we set \( \frac{dx}{dt} = 0 \), so \( 1 - x^2 - y^2 = 0 \) which is the unit circle, \( x^2 + y^2 = 1 \). To determine the h-nullclines we set \( \frac{dy}{dt} = 0 \), so \( 1 - x - y = 0 \) which is the line \( y = 1 - x \). The equilibrium points occur where the nullclines intersect, so algebraically we can substitute \( y = 1 - x \) into the v-nullcline and solve to obtain
\[
x^2 + (1 - x)^2 = 1 \implies 2x^2 - 2x = 0 \implies 2x(x - 1) = 0 \implies x = 0, 1.
\]
The corresponding y-values, using \( y = 1 - x \), are \( y = 1 \) (for \( x = 0 \)) and \( y = 0 \) (for \( x = 1 \)).
Thus the equilibrium points are \( (0, 1) \) and \( (1, 0) \).

Note that this is graphically obvious since the line’s intercepts are \( (0, 1) \) and \( (1, 0) \), both of which also lie on the unit circle.

(d) The phase portrait should look something like this (obviously your plot might not look as detailed as this one since it was made with Mathematica. Various solution trajectories have been included):
Note that the h-nullcline (grey, dotted) has leftward facing arrows on it outside of the unit circle, whereas the arrows are rightward facing inside the unit circle. To determine this, find the sign of $\frac{dx}{dt}$ using sample points such as $(-1, 2)$, $(0, 5, 0.5)$ and $(2, -1)$.

The v-nullcline (green, dotted) has upward facing arrows everywhere except the portion of the circle in the first quadrant above the line $y = 1 - x$. To determine this, note, for example, that

$$\left.\frac{dy}{dt}\right|_{(0, -1)} = 1 - 0 - (-1) = 2 > 0 \quad \text{and} \quad \left.\frac{dy}{dt}\right|_{(1/\sqrt{2}, 1/\sqrt{2})} = 1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 1 - \frac{2}{\sqrt{2}} = 1 - \sqrt{2} < 0.$$

To determine the direction of the arrows in the four regions the nullclines separate the phase plane into, note that underneath the line $y = 1 - x \frac{dy}{dt}$ must always be positive ($\frac{dy}{dt}$ is a continuous function of 2 variables, so to switch sign it must pass through zero, which occurs on the line $y = 1 - x$). We also already checked a sample point above the line, $(1/\sqrt{2}, 1/\sqrt{2})$ and determined $\frac{dy}{dt} < 0$ there, so everywhere above the line $y = 1 - x$ the arrows point downwards. By similar reasoning, outside of the circle the arrows must all be pointing leftwards since we already checked the sample point $(-1, 2)$ and the only way $\frac{dx}{dt}$ can change sign is to pass through a v-nullcline. We also know from checking the point $(0.5, 0.5)$ that arrows point rightwards inside the circle. Putting this together we see that arrows point roughly left and up outside the circle and below the line. Inside the circle and below the line they point right and up, inside the circle and above the line they point right and down, and outside the circle and above the line they point down and left.

(e) Based on the phase portrait (0, 1) is unstable whereas (1, 0) is stable.
6. (30 points) Using the Laplace transform method, solve the following initial value problem for \( y(t) \):

\[
y'' + 4y = 4\text{step}(t - 1), \quad y(0) = 1, \quad y'(0) = 0,
\]

where \( \text{step}(t) \) is the unit step function.

**Solutions:**

Taking the Laplace transform of our differential equation, we find that

\[
s^2Y(s) - sY(s) + 4Y(s) = 4\frac{e^{-s}}{s},
\]

which yields

\[
Y(s) = \frac{s}{s^2 + 4} + \frac{4e^{-s}}{s(s^2 + 4)}.
\]

Using partial fractions, we set

\[
\frac{4}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4},
\]

which gives \( A = 1 \), \( B = -1 \), \( C = 0 \). We then have

\[
\frac{4}{s(s^2 + 4)} = \frac{1}{s} - \frac{s}{s^2 + 4}.
\]

\[
Y(s) = \frac{s}{s^2 + 4} + e^{-s}\left(\frac{1}{s} - \frac{s}{s^2 + 4}\right).
\]

and

\[
y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-s}s}{s^2 + 4}\right\}.
\]

Using our table of transform pairs and the property \( \mathcal{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)\text{step}(t-c) \), we find our solution:

\[
y(t) = \cos(2t) + \text{step}(t - 1) - \text{step}(t-1) \cos(2(t - 1)).
\]
7. (30 points) After an accident, a reactor leaks a toxic substance into a container which contains some water. This material enters the container at the rate of 3 \( \mu g/\text{min} \) and mixes with the water in the container. Assume that at time \( t = 0 \), the container contains 500 \( L \) of water mixed with the toxic substance with a concentration of 1 \( \mu g/L \). In an effort to clear the container, at \( t = 0 \), engineers begin to pump pure water into the container, at a rate of 1 \( L/\text{min} \). At the same time, the container is allowed to drain into a large (clean and empty) pond at a rate of 2 \( L/\text{min} \), until the container is drained of all the solution. At this point the water pumping is stopped and the toxic leak into the container from the reactor becomes contained.

(a) (6 pts) Write down a differential equation for \( v(t) \), the volume of liquid solution in the container at time \( t \), and specify its initial condition. You can neglect the volume added by the leaking contaminant.

(b) (6 pts) Solve for \( v(t) \). At what time \( t_f \) does the container become empty of solution?

(c) (6 pts) Write down a differential equation for \( x(t) \), the amount of toxic substance in the container at time \( t \), and specify its initial condition.

(d) (12 pts) Solve for the amount of toxic substance in the container \( x(t) \) as a function of \( t \).

Solution:

(a) Let \( v(t) \) be the amount (in \( L \)) of solution in the container tank at time \( t \). At \( t = 0 \), the amount of solution in the container is 500 \( L \) so \( v(0) = 500 \). Since we can neglect the volume of the leaking contaminant, we only have a flow in of 1 \( L/\text{min} \) and a flow out of 2 \( L/\text{min} \). The differential equation and initial condition are then:

\[
\frac{dv}{dt} = 1 - 2 = -1, \quad v(0) = 500.
\]

(b) Solving the IVP we get \( v(t) = -t + 500 \), and so the container becomes empty when \( t = 500 \).

(c) At \( t = 0 \), the amount of radioactive substance in the container is 500 \( L \times 1 \mu g/L = 500 \mu g \), so \( x(0) = 500 \). The equation for \( x(t) \) is given by:

\[
\frac{dx}{dt} = 3 - 2 \frac{x(t)}{v(t)} , \quad x(0) = 500
\]

Using the expression for \( v(t) \) that we already found,

\[
\frac{dx}{dt} = 3 - 2 \frac{1}{500 - t} x,
\]

or \( x'' + \frac{2}{500-t} x = 3, \quad x(0) = 500 \)

(d) The integrating factor \( \mu(t) = e^{\int \frac{2}{500-t} dt} = e^{-2 \log |500-t|} = e^{\log |500-t|^{-2}} = |500-t|^{-2} \). We can drop the absolute value sign to take \( \mu(t) = (500 - t)^{-2} \). It follows that:

\[
\frac{d}{dt} [\mu(t)x(t)] = 3\mu(t) = 3(500 - t)^{-2} \implies \mu(t)x(t) = -3 \int u^{-2}du = 3u^{-1} + C = 3(500 - t)^{-1} + C
\]

Multiplying both sides by \((500 - t)^2\), we obtain:

\[
x(t) = 3(500 - t) + C(500 - t)^2
\]

Evaluating the initial condition \( x(0) = 500 \) yields:

\[
x(0) = 3(500) + C(500)^2 = 500 \implies C = -\frac{1000}{500^2} = -\frac{2}{500}
\]

Hence, \( x(t) = 3(500 - t) - \frac{2}{500}(500 - t)^2 \).
### TABLE OF LAPLACE TRANSFORMS

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<tr>
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<td>$\mathcal{L}(e^{at}) = \frac{1}{s-a}$, $\mathcal{L}(t^ne^{at}) = \frac{n!}{(s-a)^{n+1}}$</td>
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*IF YOU ARE FINISHED AND HAVE TIME, CHECK YOUR ANSWERS.*