Problem #1 (20 points): Answer each of the following TRUE/FALSE questions. Remember that for a statement to be TRUE, it must ALWAYS be TRUE. You do not need to justify your answer and no partial credit will be given on this problem. Each question is worth five points.

(a) [5] Consider the system of differential equations $\mathbf{x}' = A \mathbf{x}$. If $A$ is a $2 \times 2$ matrix with eigenvalues $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$, then the origin is a spiral sink.
(b) [5] The initial value problem $y' = 2y^{1/2}, y(0) = 1$ has a unique solution.
(c) [5] The solutions of $y'' - 2y' + y + \sin(t) = 0$ form a vector space (with addition and scalar multiplication having their normal definitions).
(d) [5] The dimension of $\text{span}\{e^t, e^{-t}, 2e^t + 3e^{-t}, 5e^t - 8e^{-t}\}$ is 4.

Solution: Answers to the T/F are:

(a) False, it is a spiral source.
(b) True from Picard's theorem; moreover, the unique solution is $y(t) = (1 + 3t)^{2/3}$.
(c) False. The set of all solution $\{y(t) : y(t) = (c_1 + c_2 t)e^t - \cos(t)/2\}$ doesn't contain the zero vector $y(t) = 0$ because the ODE is inhomogeneous.
(d) False, the dimension is 2 since the basis is $\{e^t, e^{-t}\}$.

Problem #2 (25 points): Find the general solution to the following differential equation:

$$y'' - 4y' + 4y = \frac{3e^{2t}}{t}, \quad t > 0.$$  

Solution: We will use variation of parameters to solve. First, we begin by solving the homogeneous problem: $y'' - 4y' + 4y = 0$. The associated characteristic equation is $r^2 - 4r + 4 = 0 \Rightarrow r = 2$. Thus,

$$y_h(t) = c_1 e^{2t} + c_2 te^{2t}$$

Now, we let $c_1$ and $c_2$ vary with time. Relabel these $v_1(t)$ and $v_2(t)$. Then, $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ where $y_1(t) = e^{2t}$ and $y_2(t) = te^{2t}$. Applying the method leads to:

$$v_1' = \frac{-te^{2t}(3e^{2t})}{te^{4t}}$$
$$v_2' = \frac{e^{2t}(3e^{2t})}{te^{4t}}$$

Simplifying and integrating yields

$$v_1 = -3t$$
$$v_2 = 3\ln(t)$$

Thus, $y_p(t) = 3te^{2t} + 3t\ln(t)e^{2t}$ and $y(t) = c_1 e^{2t} + c_2 te^{2t} + 3te^{2t} + 3t\ln(t)e^{2t}$. 

Problem #3 (20 points): Consider the linear system
\[
\begin{align*}
x_1 + x_2 + 2x_3 &= 1, \\
2x_1 - x_2 + x_3 &= 2, \\
4x_1 + x_2 + 5x_3 &= 4.
\end{align*}
\]

(a) [13] Find the associated homogeneous solution \( \bar{x}_h \).
(b) [7] Find the general solution, written in the form \( \bar{x}_h + \bar{x}_p \).

Solution: We first transform the augmented matrix to its RREF:
\[
\begin{pmatrix} A & \vert & b \end{pmatrix} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 1 & 5 & 4 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

(a) To find the associated homogeneous solution, we consider the above reduced row echelon form with the last column being replaced by zeros, that is
\[
\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \bar{x}_h = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}^T.
\]

(b) To find a particular solution, we look at the reduced row echelon form of the original augmented matrix,
\[
\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}
\]
and let the free variable \( x_3 = 1 \). Then
\[
\bar{x}_p = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}^T.
\]

Thus the general solution is
\[
\bar{x} = \bar{x}_h + \bar{x}_p = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}^T + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}^T.
\]

Problem #4 (25 points): Give a brief answer to each question. Show all your work.

(a) [10] Let
\[
D = \begin{bmatrix} 1 & 8 & -1 \\ 1 & 5 & 0 \\ 0 & 4 & -1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 5 & -1 & -4 \\ -4 & 1 & 4 \\ -5 & 1 & 3 \end{bmatrix}.
\]
Verify that \( D^T = M^{-1} \).

(b) [9] Given
\[
F = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix},
\]
find the \( 2 \times 2 \) matrix \( G \) such that \( GB = F \).

(c) [6] Let \( C \) and \( E \) be \( 2 \times 2 \) matrices with \( \det(C) = 2 \), \( \det(E) = 6 \), and \( \det(E + C) = 3 \). Find the values of
\[
\begin{align*}
\text{(i)} \quad & \det(CC^T C^{-1}(EC)^{-1}) \quad \text{and} \\
\text{(ii)} \quad & \det(4C).
\end{align*}
\]
Solution:

(a) \( D^T M = \begin{bmatrix} 1 & 1 & 0 \\ 8 & 5 & 4 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & -1 & -4 \\ -4 & 1 & 4 \\ -5 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), so \( D^T = M^{-1} \).

(b) \( G = FB^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & -16 \\ -3 & 8 \end{bmatrix} \)

(c) (i)

\[
\det(CC^T C^{-1}(EC)^{-1}) = \det(CC^T C^{-1} E^{-1}) = \det(C) \det(C^T) \det(C^{-1}) \det(E^{-1}) = 2 \cdot 2 \cdot \frac{1}{2} \cdot \frac{11}{26} = \frac{1}{6}
\]

(ii) \( \det(4C) = \det(4I \cdot C) = \det(4I) \det(C) = 4^2 \cdot 2 = 32 \)

**Problem #5 (30 points):** Consider the matrix

\[
A = \begin{bmatrix} \alpha & 0 & 0 \\ 4 & 2 & 3 \\ -5 & 0 & -1 \end{bmatrix}
\]

(a) [6] Find the eigenvalues of \( A \).

(b) [18] Find the eigenvectors of \( A \) when \( \alpha = 3 \).

(c) [6] For \( \alpha = 2 \), what is the dimension of the eigenspace associated with \( \lambda = 2 \)? (Show your work.)

Solution:

(a)

\[
\det(A - \lambda I) = \begin{vmatrix} \alpha - \lambda & 0 & 0 \\ 4 & 2 - \lambda & 3 \\ -5 & 0 & -1 - \lambda \end{vmatrix} = (\alpha - \lambda) \begin{vmatrix} 2 - \lambda & 3 \\ 0 & -1 - \lambda \end{vmatrix} = (\alpha - \lambda)(2 - \lambda)(-1 - \lambda) = 0.
\]

Therefore, \( \lambda_1 = \alpha, \lambda_2 = 2, \) and \( \lambda_3 = -1 \).

(b) For \( \alpha = 3 \),

\[
A = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 2 & 3 \\ -5 & 0 & -1 \end{bmatrix}.
\]

For \( \lambda_1 = \alpha = 3 \),

\[
A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 4 & -1 & 3 \\ -5 & 0 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4/5 \\ 0 & 1 & 1/5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = c_1 \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}.
\]

For \( \lambda_2 = 2 \),

\[
A - 2I = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 3 \\ -5 & 0 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]
For $\lambda_3 = -1$,

\[
A + I = \begin{bmatrix} 4 & 0 & 0 \\ 4 & 3 & 3 \\ -5 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_3 = c_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.
\]

(c) When $\alpha = 2$,

\[
A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 3 \\ -5 & 0 & -1 \end{bmatrix};
\]

for $\lambda = 2$,

\[
A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 3 \\ -5 & 0 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{v} = c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.
\]

Therefore, the dimension of the eigenspace associated with $\lambda = 2$ is 1.

**Problem #6 (26 points):** A block with mass 2 kg is attached to a spring with restoring constant 8 kg/s\(^2\) in an undamped system. The system is driven by the force $f(t) = 4 \sin 2t$.

(a) [14] Find the general solution to this system.

(b) [7] Describe the long-term behavior of the system.

(c) [5] Classify the equilibriums if the driving force is removed.

**Solution:**

(a) The differential equation for the system is $2\ddot{x} + 8x = 4 \sin 2t$. The associated characteristic equation is $2r^2 + 8 = 0$. Solving for $r$, you get $r = 2i, -2i$. The homogeneous solution is $x_h = C_1 \cos 2t + C_2 \sin 2t$. To find the particular solution, use MUC. Guess a solution of the form $x_p = Bt \cos 2t$. Thus

\[
\begin{align*}
\dot{x}_p &= B \cos 2t - 2Bt \sin 2t \\
\ddot{x}_p &= -4B \sin 2t - 4Bt \cos 2t
\end{align*}
\]

Substituting into the differential equation, we have

\[
2(-4B \sin 2t - 4Bt \cos 2t) + 8(Bt \cos 2t) = 4 \sin 2t
\]

\[
-8B \sin 2t = 4 \sin 2t B = -\frac{1}{2}
\]

Thus, $x_p = -\frac{1}{2} t \cos 2t$ and the general solution is $x(t) = C_1 \cos 2t + C_2 \sin 2t - \frac{1}{2} t \cos 2t$

(b) Taking the limit as $t$ approaches infinity, the oscillations increase in amplitude without bound. The system is in pure resonance.

(c) Since the eigenvalues are purely imaginary, the equilibrium is a center.

**Problem #7 (30 points):** Consider the nonlinear predator-prey-like system

\[
\begin{align*}
x' &= (x - \mu)x - xy, \\
y' &= (x - 1)y,
\end{align*}
\]

where $\mu > 0$, $x \geq 0$, and $y \geq 0$.

(a) [5] Find all equilibrium point(s).

(b) [5] Calculate the linear stability of all of the equilibrium point(s) you found.

(c) [5] Does the number of equilibrium points depend on the value of $\mu$? If so, explain why.
(d) [5] Assume $\mu = 0.5$. Plot the nullclines in the phase plane and draw arrows indicating the direction that trajectories through the nullclines would take.

(e) [5] Label the equilibrium points

(f) [5] Sketch some sample trajectories.

Solution:

(a) Solving $x' = 0$ yields the x-nullclines

\[
\begin{align*}
x &= 0, \quad \text{and} \\
y &= x - \mu
\end{align*}
\]

Similarly, we find the y-nullclines

\[
\begin{align*}
x &= 1, \quad \text{and} \\
y &= 0
\end{align*}
\]

The equilibriums occur where the nullclines cross. We have three equilibrium points: $(0,0)$, $(\mu, 0)$ and $(1, 1 - \mu)$.

(b) We calculate the Jacobian of the system to be:

\[
J = \begin{bmatrix}
2x - (\mu + y) & -x \\
y & x - 1
\end{bmatrix}
\]

Evaluating $J$ at each equilibrium point gives us:

\[
J_{(0,0)} = \begin{bmatrix}
-\mu & 0 \\
0 & -1
\end{bmatrix}, \quad J_{(\mu,0)} = \begin{bmatrix}
\mu & -\mu \\
0 & \mu - 1
\end{bmatrix}, \quad J_{(1,1-\mu)} = \begin{bmatrix}
1 & -1 \\
1 - \mu & 0
\end{bmatrix}
\]

From the trace and determinant of these matrices, we can see (since $\mu > 0$) that $(0, 0)$ is stable, and the other equilibrium points are unstable.

(c) From the given information, we see that $(1, 1 - \mu)$ is a relevant equilibrium point only if $\mu < 1$. 

(d) $x' = (x - \mu) x - x y$  
$y' = (x - 1) y$  
$\mu = 0.5$
Problem #8 (24 points): Give a short answer to each question. Box your answer. No work for this question will be graded.

(a) [6] Does Picard’s Theorem apply to $y' = y^{4/3}$, $y(0) = 0$? Briefly explain why or why not.
(b) [6] Consider the system

$$x' = kx,$$
$$y' = -y.$$ 

The stability of the system’s solutions depends on the constant $k$. For all possible values of $k$, classify the equilibrium solutions’ stability.

(c) [6] A 100 liter tank initially contains 50 liters of water with 42 grams of kool-aid powder dissolved in the water. A solution with a kool-aid concentration of 1 gram/liter enters the tank at a rate of 10 liters/hour. If a well-mixed solution leaves the tank at a rate of 9 liters/hour, what is the initial value problem that models the amount of kool-aid in the tank?

(d) [6] Which of the following equations have a stable equilibrium solution at $y(t) = 1$:

(a) $y' = -y$, (b) $y' = y(y-1)$, (c) $y' = (y-1)(y-2)$.

Solution: The solutions are:

(a) Yes, Picard’s theorem applies as for $f(t, y) = y^{4/3}$, both $f$ and $f_y$ are continuous at $y = 0$.

(b) As $k$ changes from negative to positive, the stability of the equilibrium solution changes from stable to unstable.

(c) The initial value problem is $x' = 10 - \frac{9x}{50 + t}$; $x(0) = 42$.

(d) Only (c) has a stable equilibrium at $y(t) = 1$. 

---

6