Problem 1: (20 points) For the questions in this problem, no motivation is required. If you do submit work, then box your answer, and know that your work will not be graded.

(a) (5 points) Two matrices $A$ and $B$ have inverses:

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$$

Use these to find the unique solution $x$ to the system $(AB)x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) (5 points) What is the basis and dimension of the subspace $W$ of $\mathbb{R}^3$ where $W = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$?

(c) (5 points) Consider the matrix $A = \begin{bmatrix} 0 & 1 & 5 \\ 1 & c & 4 \\ 0 & 0 & -1 \end{bmatrix}$. For what real values of $c$ is $A$ invertible?

(d) (5 points) Consider the two matrices $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Determine the value of

(i) $\det(A)$
(ii) $\det(B)$
(iii) $\det(A^TB^{-1}A^T)$

Solution:

(a) 

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 7 & 8 \end{bmatrix}$$

So the solution $x$ is given by:

$$x = (AB)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 26 \\ 23 \end{bmatrix}$$

(b) With the introduction of two free variables, $W$ is spanned by two basis vectors. Hence, $\dim(W) = 2$. Two basis vectors can be $v_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Note that the equation defines a plane through the origin in $\mathbb{R}^3$. This has dimension 2.

(c) Compute the determinant via a cofactor expansion along the third row: $\det A = \begin{vmatrix} 1 & 5 \\ 1 & 4 \\ 0 & c \end{vmatrix} = 1$. The matrix $A$ is invertible for any value of $c$.

(d) Use properties of the determinant:

(i) $\det A = 2 \cdot 2 \cdot 1 = 4$

(ii) $\det B = 1 \cdot 2 \cdot 3 = 6$

(iii) $\det(A^TB^{-1}A^T) = \det(A^T)\det(B^{-1})\det(A^T) = \det(A)\det(B)^{-1}\det(A) = 4 \cdot 6^{-1} \cdot 4 = 8/3$.
Problem 2: (15 points) For the following True/False questions, just state True or False. Your work will not be graded.

(a) (3 points) Let $A$ be a $n \times n$ invertible matrix. Then $A^2$ is invertible.

(b) (3 points) Let $A$ be a $3 \times 5$ matrix then the dimension of the space $\text{null } A = \{x \mid Ax = 0\}$ is less than or equal to 3.

(c) (3 points) Let $\{v_1, v_2, v_3, v_4\}$ be vectors in $\mathbb{R}^3$. Then $\text{span } \{v_1, v_2, v_3, v_4\} \neq \mathbb{R}^3$.

(d) (3 points) Let $A$ and $B$ be matrices. If $AB = B$, then $A$ must be an identity matrix.

(e) (3 points) Let $A$ be a $3 \times 3$ matrix. Suppose $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has a unique solution $x$, then $A$ is invertible.

Solution:

(a) $(A^2)^{-1} = (AA)^{-1} = A^{-1}A^{-1} = (A^{-1})^2$. True.

(b) $Ax = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$ so $x \in \mathbb{R}^5$. Let $A$ be a $3 \times 5$ matrix of zeros, then $\text{null } A = \mathbb{R}^5$ so $\dim \text{null } A = 5$. False.

(c) Choose for the four vectors any basis for $\mathbb{R}^3$ (that’s three of them) and add any vector from $\mathbb{R}^3$ (that’s four total). Since the basis already spans the entire space, we have $\text{span } \{v_1, v_2, v_3, v_4\} = \mathbb{R}^3$. Adding another vector to a span will not remove any elements from the resultant space. False.

(d) Let $A$ be any $n \times n$ matrix. Take $B$ to be a $n \times m$ matrix of zeros. Then $AB = B$. False.

(e) Suppose $A$ is not invertible. Then there is a nonzero $y$ such that $Ay = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. But then $A(x + y) = Ax + Ay = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ so we have produced another solution $x + y$ contradicting the uniqueness of $x$. True.
Problem 3: (20 points)
Let \( M \) be the matrix:
\[
M = \begin{bmatrix}
1 & -2 & 5 & -3 \\
2 & 3 & 1 & -4 \\
3 & 8 & -3 & -5
\end{bmatrix}
\]

(a) (2 points) What is the rank of \( M \)?
(b) (6 points) Find a basis and the dimension of the set \( V = \) row space of \( M \).
(c) (6 points) Find a basis and the dimension of the set \( W = \) column space of \( M \).
(d) (6 points) Find a basis and the dimension of the set \( Z = \) null space of \( M \).

Solution:

We form \( M \) and row reduce to echelon form (reduction to reduced echelon form is not necessary).

\[
M = \begin{bmatrix}
1 & -2 & 5 & -3 \\
2 & 3 & 1 & -4 \\
3 & 8 & -3 & -5
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -2 & 5 & -3 \\
0 & 1 & -9 & 9 \\
0 & 1 & -9 & 9
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -2 & 5 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(a) \( \text{rank} M = 2 \) (two linearly independent rows).
(b) The linearly independent vectors \( \mathbf{r}_1 = [1 \ -2 \ 5 \ -3] \) and \( \mathbf{r}_2 = [0 \ 1 \ -9/7 \ 2/7] \) span the row space of \( M \), hence form a basis. \( \text{dim row} \ M = 2 \).
(c) The linearly independent vectors \( \mathbf{u}_1 = [1 \ 2 \ 3] \) and \( \mathbf{u}_2 = [-2 \ 3 \ 8] \) (or \( \mathbf{u}_1 \) and any other column of \( M \)) are a basis for the column space of \( M \). \( \text{dim col} \ M = 2 \).
(d) We need to find all the solutions to \( M\mathbf{x} = 0 \). Reduce \( M \) to RREF starting where we left off above:

\[
M \rightarrow \begin{bmatrix}
1 & 0 & \frac{17}{7} & -\frac{17}{7} \\
0 & 1 & -\frac{9}{7} & \frac{2}{7} \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Now use the first two rows of RREF. Suppose \( \mathbf{x} \in \text{null} \ A \), then
\[
x_1 + \frac{17}{7}x_3 - \frac{17}{7}x_4 = 0,
\]
\[
x_2 - \frac{9}{7}x_3 + \frac{2}{7}x_4 = 0.
\]

Letting \( x_3 = s \), \( x_4 = t \), \( s, t \in \mathbb{R} \), we can write \( \mathbf{x} = s \begin{bmatrix} -\frac{17}{7} \\ \frac{9}{7} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{17}{7} \\ 0 \\ 0 \\ 1 \end{bmatrix} \) so \( \text{dim null} \ A = 2 \).

and a basis is \[ S = \left\{ \begin{bmatrix} -\frac{17}{7} \\ \frac{9}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{17}{7} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \]
Problem 4: (15 points)

Given the matrix: $$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) (10 points) Find all solutions to the linear system of equations $Ax = b$ where $b = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$.

(b) (5 points) Determine $\text{RREF } A$.

Solution:

First, we derive the RREF:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 1 & 2 & 4 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 2 & 2 & 2 & -2 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) $x_3$ is the free variable, set $x_3 = 0$ we solve the reduced equation to get:

$$x_p = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

and set $x_3 = 1$ for the homogeneous equation to get $x_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

$$x_g = x_p + x_h = x_p + cx_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ for any } c \in \mathbb{R}.$$
Problem 5: (20 points) Determine whether or not the following sets \( \mathbb{W} \) are vector subspaces of \( \mathbb{V} \). For each set, provide a yes or no answer and, to receive full credit, you must explain why it is or is not a vector subspace. You may assume that all the sets \( \mathbb{V} \) are vector spaces.

(a) (5 points) Given a square \( n \times n \) matrix \( A \in \mathbb{M}_{nn} \), the set of vectors
\[
\mathbb{W} = \{ u \mid Au = u \} \subset \mathbb{V} = \mathbb{R}^n.
\]

(b) (5 points) The set of \( 2 \times 2 \) determinant zero matrices
\[
\mathbb{W} = \{ A \mid \det A = 0 \} \subset \mathbb{V} = \mathbb{M}_{22}.
\]

(c) (5 points) The solution set
\[
\mathbb{W} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x^2 + y^2 + z^2 = 0 \right\} \subset \mathbb{V} = \mathbb{R}^3.
\]

(d) (5 points) The set of all solutions to the differential equation
\[
\mathbb{W} = \left\{ y(t) \mid \ddot{y} + \frac{e^t}{t} \dot{y} - y = 0 \right\} \subset \mathbb{V} = C^2([1, \infty)).
\]

**Solution:**

In order to be a vector subspace of a known vector space, we must check if the set is closed under vector addition and scalar multiplication (the subspace theorem).

(a) **Yes** Let \( u, v \in \mathbb{V} \) and \( \alpha \in \mathbb{R} \), then we verify closure
\[
A(u + v) = Au + Av
\]
\[= u + v \implies u + v \in \mathbb{V},\]

\[
A(\alpha u) = \alpha Au
\]
\[= \alpha u \implies \alpha u \in \mathbb{V}.
\]

(b) **No** The determinant is a linear operator on rows, not on the entire matrix. We need a counterexample. Here is one.

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]

\[
|A| = 0, \quad |B| = 0 \implies A, B \in \mathbb{V}.
\]

\[
|A + B| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \implies A + B \notin \mathbb{V}.
\]

(c) **Yes** \( V = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \), which is always a vector space.

(d) **Yes** The differential equation is linear and homogeneous so the solution set is closed under addition (the superposition principle) and under scalar multiplication (linearity).
Problem 6: (10 points)

Let \( A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \). Compute \( A^{-1} \).

Solution:

The inverse of \( A \) can be computed by Gaussian elimination:

\[
[A|I] = \begin{pmatrix}
1 & -1 & 1 & | & 1 & 0 & 0 \\
0 & 1 & -1 & | & 0 & 1 & 0 \\
0 & 0 & 1 & | & 0 & 0 & 1
\end{pmatrix}
\]

\[
R_1 \rightarrow R_1 + R_2 \\
\begin{pmatrix}
1 & 0 & 0 & | & 1 & 1 & 0 \\
0 & 1 & -1 & | & 0 & 1 & 0 \\
0 & 0 & 1 & | & 0 & 0 & 1
\end{pmatrix}
\]

\[
R_2 \rightarrow R_2 + R_3 \\
\begin{pmatrix}
1 & 0 & 0 & | & 1 & 1 & 0 \\
0 & 1 & 0 & | & 0 & 1 & 1 \\
0 & 0 & 1 & | & 0 & 0 & 1
\end{pmatrix}
\]

Therefore

\[
A^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\]