1. [36 points] Answer each of the following TRUE/FALSE questions. Remember that for a statement to be TRUE, it must ALWAYS be TRUE. You do not need to justify your answer and no partial credit will be given on this problem. Each question is worth six points.

(a) There exists a real $2 \times 2$ matrix, $A$, with eigenvalues $\lambda_1 = 1 + i$, $\lambda_2 = -1 + i$.
(b) If $\lambda = 2$ is a repeated eigenvalue of multiplicity 3 for a matrix $A$, then the eigenspace associated with $\lambda$ has dimension 3.
(c) If $y_1(t)$ and $y_2(t)$ are two solutions of $y'' + p(t)y' + q(t)y = 0$, then the formula $y(t) = c_1y_1(t) + c_2y_2(t)$ gives all solutions to $y'' + p(t)y' + q(t)y = 0$.
(d) If $A$ is an $n \times n$ matrix and $\det(A) \neq 0$ then $Ax = b$ has a unique solution for any $b \in \mathbb{R}^n$.

Solution: Answers to the T/F are:

(a) False. Complex eigenvalues of a real matrix must occur in complex conjugate pairs.
(b) False. The matrix could be degenerate and $\lambda = 2$ could have geometric multiplicity less than 3.
(c) False. The solutions $y_1(t)$ and $y_2(t)$ could be linearly dependent and thus a linear combination will not provide all solutions to the second order differential equation.
(d) True. When the determinant is non-zero, the matrix is invertible.
(e) True. These two solutions are linearly independent. Both solve this second order equation and thus span the solution space.
(f) True.

2. [35 points] Using any appropriate method:

(a) (18 points) solve the initial value problem

$$\frac{dy}{dx} = \frac{y}{x+y}, \quad y(x = 0) = 1.$$  \hspace{1cm} (1)

Hint: Treat $x$ as a function of $y$ (remember that $dy/dx = (dx/dy)^{-1}$). No need to solve for $y$ in terms of $x$ in the final result.

(b) (17 points) solve the initial value problem

$$\frac{d^2y}{dx^2} - y = e^x, \quad y(x = 0) = 0, \quad y'(x = 0) = \frac{1}{2}.$$  \hspace{1cm} (2)

Solution:

(a) Treating $x$ as a function of $y$, we have

$$\frac{dx}{dy} - \frac{x}{y} = 1, \quad x(y = 1) = 0.$$  \hspace{1cm} (3)

An integrating factor is

$$u = \exp\left(-\int \frac{1}{y}dy\right) = \exp(-\log y) = \frac{1}{y}.$$  \hspace{1cm} (4)
so we have
\[
\frac{d}{dy} \left( \frac{x}{y} \right) = \frac{1}{y},
\] (5)
which may be integrated to yield
\[
\frac{x}{y} = \log y + c,
\] (6)
c being a constant. The initial condition requires
\[
c = 0,
\] (7)
so the solution to the IVP is
\[
x = y \log y.
\] (8)

(b) The homogeneous ODE
\[
\frac{d^2 y}{dx^2} - y = 0
\] (9)
has the characteristic equation
\[
r^2 - 1 = 0 \quad \Rightarrow \quad r = \pm 1,
\] (10)
and hence the general solution
\[
y_h(x) = c_1 e^x + c_2 e^{-x}.
\] (11)
A particular solution \( y_p \) may be determined using MUC. Since the forcing function \( e^x \) is inside the vector space of \( y_h \), we guess
\[
y_p = a x e^x.
\] (12)
Since \( y'_p = a e^x (x + 1) \), \( y''_p = ae^x (x + 2) \), substitution into the original ODE yields
\[
2a = 1 \quad \Rightarrow \quad a = \frac{1}{2}.
\] (13)
The general solution is therefore
\[
y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x.
\] (14)
Since \( y'(x) = c_1 e^x - c_2 e^{-x} + \frac{1}{2} e^x (x + 1) \), the initial condition requires
\[
c_1 + c_2 = 0, \quad c_1 - c_2 = 0 \quad \Rightarrow \quad c_1 = 0, \quad c_2 = 0,
\] (15)
so the solution to the IVP is
\[
y(x) = \frac{1}{2} x e^x.
\] (16)

3. [36 points]

(a) (14 points) Consider the following matrix and vector
\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
-1 & -1 & -1
\end{bmatrix} \quad \vec{b} = \begin{pmatrix}
1 \\
-1 \\
-1
\end{pmatrix}
\]
i. Compute the general solution to \( A \vec{x} = \vec{b} \)
ii. What is the basis for the solutions space to \( A \vec{x} = 0 \)?

(b) (8 points) For \( c \) a real number, consider the matrix
\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & c \\
0 & 0 & 0 & 0 & -c & 1
\end{bmatrix}.
\]
For an arbitrary value of \( c \), what is the range of values that \( |C| \) could be?
(c) (14 points) Compute the inverse of the following matrix

\[
D = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & d \\
0 & -d & 0
\end{bmatrix},
\]

and specify the values of \( d \) that allow an inverse.

**Solution:**

(a) The augmented matrix is

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 0 & -1 \\
-1 & -1 & -1 & -1
\end{bmatrix}
\]

and the RREF process yields

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & -1 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

i. Thus the general solution is

\[
\vec{x} = s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}
\]

for any value of \( s \).

ii. And the basis for the solution space to \( A\vec{x} = \vec{0} \) is

\[
\vec{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}
\]

(b) By remembering the algorithm to compute a determinant, we have that

\[|C| = 1 + c^2\]

and thus \(|C| \geq 1\).

(c) The RREF process yields

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & d & 0 & 1 & 0 \\
0 & -d & 0 & 0 & 0 & 1
\end{bmatrix}
\]

and thus the inverse is

\[
D^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & -1/d \\
0 & 1/d & 1/d^2
\end{bmatrix},
\]

where \( d \neq 0 \) allows for an inverse.

4. [35 points] Consider the matrix

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}.
\]

(a) (10 points) Find all the eigenvalues of \( A \).
(b) (15 points) For each eigenvalue that you found in part (a), find a basis for the corresponding eigenspace.

(c) (10 points) Find the general solution of \( x'' = Ax \). \textbf{(Hint:} you need to find a generalized eigenvector\textbf{)}

\textbf{Solution:}

(a) The characteristic polynomial of \( A \) is \( p(\lambda) = (1 - \lambda)^2(2 - \lambda) \). The eigenvalues are \( \lambda_1 = 1 \) (repeated) and \( \lambda_2 = 2 \).

(b) \( \lambda_1 = 1 \) Solve \((A - \lambda_1 I)\vec{v}_1 = 0:\)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.
\]

Then \( \vec{v}_1 \) is a basis for the eigenspace \( E_1 \) corresponding to \( \lambda_1 = 1 \).

\( \lambda_2 = 2 \) Solve \((A - \lambda_2 I)\vec{v}_2 = 0:\)

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
2 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\Rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.
\]

So \( \vec{v}_2 \) is a basis for the eigenspace \( E_2 \) corresponding to \( \lambda_2 = 2 \).

(c) \( x'' = Ax \) is a \( 3 \times 3 \) system, so we need three linearly independent solutions. Two solutions are given by \( \vec{x}_1(t) = e^{\lambda_1 t} \vec{v}_1 \) and \( \vec{x}_2(t) = e^{\lambda_2 t} \vec{v}_2 \). For the third, we need to find a generalized eigenvector for the eigenvalue \( \lambda_1 = 1 \), since this is the repeated eigenvalue.

Solve \((A - \lambda_1 I)\vec{u} = \vec{v}_1:\)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\Rightarrow \vec{u} = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

Then a third linearly independent solution is given by

\[
\vec{x}_3(t) = te^{\lambda_1 t} \vec{v}_1 + e^{\lambda_1 t} \vec{u} = te^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}.
\]

The general solution is given by

\[
\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
\]

\[
= c_1 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{t} \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}.
\]

5. [36 points] Let \( u'' + bu' + 25u = A \cos(\omega_f t) \).

(a) (12 points) Consider the forced system with \( b = 0 \) and \( A = 100 \). Write the general solution \( u_G \) for

i. (i) when \( \omega_f \) does not equal the resonant frequency

ii. (ii) when \( \omega_f \) does equal the resonant frequency.

(b) (6 points) Use your solution from part (a) for \( \omega_f = \sqrt{50} \) to solve the IVP for \( u(0) = 1 \) and \( u'(0) = 50 \).

(c) (12 points) Find the general solution with \( b = 6 \), \( A = 1 \), and \( \omega_f = 1 \).

(d) (6 points) Use your solution from part (c) to solve the IVP with \( u(0) = u'(0) = 0 \).

\textbf{Solution:}

(a) When \( \omega_f \neq \omega_0 \), one has

\[
u_G = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{A}{\omega_0^2 - \omega_f^2} \cos(\omega_f t)
\]

\[
= C_1 \cos(5t) + C_2 \sin(5t) + \frac{100}{25 - \omega_f^2} \cos(\omega_f t).
\]
6. [36 points] Consider the following system of equations ONLY for \( x \geq 0 \) and \( y \geq 0 \), i.e., in the first quadrant.

\[
\begin{align*}
x' &= -x(y - x) \\
y' &= y(x - 1).
\end{align*}
\]
(a) (30 points) Draw and label the phase plane, including:

i. All equilibrium points (be sure to identify their stability).
ii. All nullclines (with arrows indicating solution direction).
iii. Two example solution curves.

(b) (6 points) What is the long-term behavior for a solution with initial condition of \((x(0), y(0)) = (1, 10^{-2360})\)

Solution:

(a) The phase plane is depicted in the following figure

\[
\begin{align*}
\dot{x} &= -x (y - x) \\
\dot{y} &= y (x - 1)
\end{align*}
\]

i. The equilibrium points in the first quadrant are at \((0,0)\) and \((1,1)\). The Jacobian is

\[
J(x, y) = \begin{bmatrix} -y + 2x & -x \\ y & x - 1 \end{bmatrix}.
\]

For \((0,0)\), \(\text{Tr} J(0,0) = -1\) and \(|J(0,0)| = 0\) and so the node at \((0,0)\) is an attracting line of equilibrium points (attracted to the x-axis). For \((1,1)\), \(\text{Tr} J(1,1) = 1\), \(|J(1,1)| = 1\), and \(\Delta = 1^2 - 4 \cdot 1 < 0\) and so the node is a repelling spiral.

ii. The \(x'\) nullcline is \(x = 0\) and this line \(y = x\) while the \(y'\) nullcline is \(y = 0\) and \(x = 1\). See the phase plane figure for the direction of the solutions.

(b) The long term behavior of \((1, 10^{-2360})\) tends toward \((\infty, \infty)\).

7. [36 points] Give a short answer to each question. Box your answer. No work for this question will be graded. Each problem is worth 9 points.

(a) Consider the initial value problem

\[
\begin{align*}
y' &= y^2 \\
y(1) &= 1
\end{align*}
\]

For what value of \(t\) does the solution not exist?
(b) Suppose \( y_1(x) = x, \ y_2(x) = x^2 \) and \( y_3(x) = x^3 \) are all solutions to the second-order differential equation

\[
\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x).
\]

(18)

Write down the general solution to this differential equation without knowing explicitly the functions \( p(x), q(x) \) and \( f(x) \).

(c) Consider the two-dimensional linear system \( \mathbf{x}' = A\mathbf{x} \) where

\[
A = \begin{bmatrix} a & 1 \\ -2 & 1 \end{bmatrix}.
\]

For what value(s) of \( a \) does the stability of the equilibrium \( \mathbf{x} = (0,0) \) change?

(d) Let \( \mathcal{V} \) be the space of \( 3 \times 3 \) matrices with real values and let \( \mathcal{W} = \{ A \in \mathbb{R}^{3 \times 3} : A^T = -A \} \). Is \( \mathcal{W} \) a subspace of \( \mathcal{V} \)?

Solution:

(a) By using separation of variables and applying the initial condition, the solution is \( y(t) = (2 - t)^{-1} \).

Thus at \( t = 2 \), the solution is undefined.

(b) The differences between two particular solutions \( y_p \), say \( y_{h1} = y_1 - y_2 = x - x^2 \) and \( y_{h2} = y_2 - y_3 = x^2 - x^3 \), are homogeneous solutions due to linearity. Since \( y_{h1} \) and \( y_{h2} \) are linearly independent, they span the vector space of homogeneous solutions. The general solution may therefore be written in various ways, e.g.

\[
y = y_h + y_p = c_1(x - x^2) + c_2(x^2 - x^3) + x.
\]

(c) Since \( |A| = a + 2 \) and \( \text{tr}A = a + 1 \), the boundaries for stability change on the \( (\text{tr}A, |A|) \)-plane are crossed at \( a = -1 \) or \( a = -2 \).

(d) Consider two matrices \( B \) and \( C \) such that \( B^T = -B \) and \( C^T = -C \). Also consider another matrix \( D = \alpha B + \beta C \) where \( \alpha, \beta \) are real. Now note that

\[
D^T = (\alpha B + \beta C)^T
= \alpha B^T + \beta C^T
= -\alpha B - \beta C
= -D.
\]

This proves that \( D \) is in \( \mathcal{W} \) and thus \( \mathcal{W} \) is a subspace of \( \mathcal{V} \).