ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) recitation section, (4) your instructor’s name, and (5) a grading table. Books, class notes, cell phones, and calculators are NOT permitted. A one-page one-sided crib sheet is allowed.

1. (25 points, Full Answer Problem): Partial credit given, please show all work.
   a. (10 points) Using separation of variables, solve the following equation to find \( y(t) \):
   \[
   \frac{dy}{dt} = 2ty^2.
   \]
   (1)
   b. (5 points) Verify your solution to part (a) by substituting your expression for \( y(t) \) back into Equation (1).
   c. (10 points) Suppose that we seek a solution to Equation (1) with the initial condition \( y(0) = y_0 \). Using this information to eliminate the constant of integration, rewrite your expression from part (a) in terms of \( y_0 \).

   **Solution:**
   a. First, note that \( y = 0 \) is an equilibrium solution of the ODE. For other solutions, separating variables yields
   \[
   \frac{dy}{y^2} = 2tdt \Rightarrow \frac{1}{y} = -t^2 + C \Rightarrow y = \frac{1}{C - t^2},
   \]
   where \( C \) is an undetermined constant of integration.
   b. Differentiating our expression for \( y(t) \), we find that
   \[
   \frac{dy}{dt} = \frac{2t}{(C - t^2)^2}(-2t) = \frac{2t}{(C - t^2)^2} = 2ty^2
   \]
   (3)
   c. Substituting in the values \( t = 0, y = y_0 \) into our expression for \( y(t) \), we find that
   \[
   y_0 = \frac{1}{C - 0^2} = \frac{1}{C} \Rightarrow C = \frac{1}{y_0},
   \]
   which yields
   \[
   y(t) = \frac{1}{\frac{1}{y_0} - t^2}
   \]
   (5)

2. (25 points, Full Answer Problem): Partial credit given, please show all work.
   a. (20 points) Find the general solution to
   \[
   t^2\frac{dy}{dt} - y = 1
   \]
   using the integrating factor method.
   b. (5 points) If \( y_1 \) and \( y_2 \) are both solutions to \( y' + 2y = -t \), then is \( y_3 = y_1 + y_2 \) also a solution to \( y' + 2y = -t \)?

   **Solution:**
   a. (20 points) This is the same as solving \( t^2y' - y = 1 \), which we re-write into the form \( y' - \frac{y}{t^2} = \frac{1}{t^2} \) before applying the method of integrating factors. Then find the integrating factor \( \mu \), either via formula or by the consideration that we want \( \mu y' - \mu \frac{y}{t^2} = \frac{d}{dt}(\mu y) = \mu' y + \mu y' \). This gives the ODE for \( \mu \):
   \[
   \mu' = \frac{\mu}{t^2}
   \]
which is solved by (use separation of variables) \(\ln |\mu| = -\int \frac{dt}{t^2} = \frac{1}{t} + c\) (and we can ignore the constant). Hence \(\mu(t) = e^{1/t}\) will work as the integrating factor.

Then \(\frac{d}{dt}(\mu y) = \frac{\mu}{t^2}\), so we plug in the value of \(\mu\) and integrate this to get

\[
\mu y = \int \frac{e^{1/t}}{t^2} dt.
\]

This integral can be solved with a \(u\)-substitution, letting \(u = 1/t\) so \(du/dt = -1/t^2\), hence

\[
\int \frac{e^{1/t}}{t^2} dt = \int \frac{e^u}{t^2} \cdot -t^2 du = -e^u + c = -e^{1/t} + c.
\]

Thus

\[
y(t) = \frac{1}{\mu(t)} \left(-e^{1/t} + c\right) = e^{-1/t} \left(-e^{1/t} + c\right) = -1 + ce^{-1/t}.
\]

We can verify that \(t^2 y' - y = 1\) by calculating \(y' = \frac{c}{t^2} e^{-1/t}\) so

\[
t^2 \left(\frac{c}{t^2} e^{-1/t}\right) - \left(-1 + ce^{-1/t}\right) = 1.
\]

b. (5 points) If \(y_1\) and \(y_2\) are both solutions to \(y' + 2y = -t\), then is \(y_3 = y_1 + y_2\) also a solution to \(y' + 2y = -t\)? [No] The equation is linear but not homogeneous.

3. (25 points, Full Answer Problem): Partial credit given, please show all work.

a. (20 points) Find the general solution of the ODE

\[
\frac{dy}{dt} = t - y
\]

using the method of variation of parameters (i.e., the Euler-Lagrange method).

b. (5 points) Solve the IVP with the above differential equation and the initial condition \(y(0) = 0\).

\textbf{Solution:}

a. Rewrite the equation in standard first order linear equation form as \(y' + y = t\). The corresponding homogeneous equation satisfies \(y'_h + y_h = 0 \implies y_h(t) = Ce^{-t}\). Set \(y_p(t) = v(t)e^{-t}\) and plug into the non-homogeneous equation: \(y'_p + y_p = t\). Then \(v'e^{-t} = t \implies v' = te^t\). Using IBP, set \(u = t, dv = e^t dt \implies v = e^t\) to get \(v(t) = te^t - e^t = (t-1)e^t\). Thus, \(y_p(t) = (t-1)e^t e^{-t} = (t-1)\) and the general solution is: \(y(t) = y_h(t) + y_p(t) = Ce^{-t} + (t-1)\).

b. Plugging in the IC, we get \(y(0) = C - 1 = 0 \implies C = 1\). So that \(y(t) = e^{-t} + (t-1)\).

\textbf{TURN OVER}
4. **(40 points, Short Answer Problem):** For the questions in this problem, no motivation is required. If you do submit work, then box your answer, and know that your work will not be graded.

a. **(10 points)** Consider the initial value problem

\[ y' = y(1 - y)(3 - y), \quad y(0) = 2. \]

Using qualitative analysis, find \( \lim_{t \to \infty} y(t) \).

b. **(10 points)** The plots in the figure below show the direction fields of the differential equations (i) \( y' = yt \), (ii) \( y' = y + t \), and (iii) \( y' = y - t \) (not necessarily in that order). Which differential equation (i, ii, iii) goes with which plot (a, b, c)?

![Direction field plots](image)

C. **(10 points)** The number of bacteria in a colony increases at a rate proportional to the number present. If the number of bacteria doubles in 8 hours, how long will it take for the colony to triple in size?

d. **(10 points)** Consider the IVP \( y' = y \), \( y(0) = 1 \). Find \( y(1) \). What does Euler’s method give for the approximation \( y_1 \) of \( y(1) \), after one step with step size \( h = 1 \)?

**Solution:**

a. Notice that \( y = 1 \) is an equilibrium solution. For \( 1 < y < 3 \), the sign of \( y' \) is negative so a solution starting at \( y(0) = 2 \) would converge down to the equilibrium solution \( y = 1 \). Hence, the limit is 1.

b. Look at the given ODEs and the slope fields, identify key properties of each equation: (e.g. \( y' = y + t \) has slope zero along the line \( y = -t \); while \( y' = y - t \) has slope zero along the line \( y = t \); \( y' = yt \) has an equilibrium solution \( y = 0 \). These alone are enough to deduce the answer). The corresponding matches are (b), (a), (c).

c. Since the rate of growth is proportional to the number present, the differential equation modeling the behavior is:

\[ \frac{dA}{dt} = kA \implies A(t) = A_0 e^{kt} \quad \text{so} \quad 2A_0 = A_0 e^{8k} \implies \ln(2) = 8k \implies k = \frac{\ln(2)}{8}. \]

Now solve

\[ 3A_0 = A_0 e^{(\ln(2)/8)t} \implies t = \frac{8 \ln(3)}{\ln(2)} \]

d. Solution to IVP is \( y(x) = e^x \). Hence, \( y(1) = e \approx 2.72 \). One step of Euler’s method with \( f(x, y) = y \), gives \( y_1 = y_0 + hf(0, 1) = 1 + (1)(1) = 2 \).

5. **(35 points, Full Answer Problem):** Partial credit given, please show all work. A tank has a capacity of 50 gal. At the start of an experiment, 4 lb of salt are dissolved in 20 gal of water
in the tank. A salt solution with concentration 2 lb/gal is added at a rate of 3 gal/min, and the well-stirred mixture is drained out at the same rate of flow.

a. (15 points) Write down and solve an IVP to find the amount of salt in the tank as a function of time.

b. (6 points) What will be the limiting concentration of salt in the tank (as \( t \to \infty \))? You must use your answer from part (a). Interpret your answer in one sentence.

c. (6 points) How long should this process continue to raise the amount of salt in the tank to 30 lb?

d. (8 points) Now suppose that, at the instant the amount of salt in the tank reaches 30 lbs, the salt solution is turned off and pure water is added to the tank at a rate of 5 gal/min. The well-stirred mixture is drained from the tank at the original rate of 3 gal/min. Write down the new IVP for the rate of change of the amount of salt in the tank (you do not need to solve the IVP).

**Solution:**

a. Let \( x(t) \) be the number of lbs of salt in the tank at time \( t \) (in min). Then

\[
\frac{dx}{dt} = (2 \text{ lb/gal})(3 \text{ gal/min}) - \left( \frac{x(t)}{20} \right) (3 \text{ gal/min}), \quad x(0) = 4
\]

is our IVP. But \( \frac{dx}{dt} = 6 - \frac{3}{20} x \) is first-order linear with constant coefficients, so \( x(t) = x_h(t) + x_p(t) \) where \( x_h(t) = Ce^{-3t/20} \) is the general solution of the homogeneous version of the diffeq \( \frac{dx}{dt} = -\frac{3}{20} x \) and \( x_p(t) = 40 \) is the equilibrium solution of the original nonhomogeneous diffeq. Thus \( x(t) = Ce^{-3t/20} + 40 \) is the general solution of the diffeq. Using \( x(0) = 4 \) we can solve for \( C \):

\[
4 = Ce^0 + 40 \quad \implies \quad C = -36
\]

is the amount of salt in the tank at time \( t \).

b. Note that

\[
\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \left( -36e^{-3t/20} + 40 \right) = 0 + 40 = 40,
\]

so the limiting concentration of salt in the tank is \( 40/20 = 2 \text{ lbs/gal} \). This makes sense since this is the concentration of the solution entering the tank.

c. Solve \( x(t) = 30 \) for \( t \):

\[
30 = -36e^{-3t/20} + 40 \iff -10 = -36e^{-3t/20} \iff \frac{5}{18} = e^{-3t/20} \iff \ln \left( \frac{5}{18} \right) = -3t/20
\]

\[
\iff t = \frac{-20}{3} \ln \left( \frac{5}{18} \right) \approx 8.54 \text{ min}
\]

d. We have a new IVP:

\[
\frac{dx}{dt} = 0 - \left( \frac{x(t)}{20 + 2t} \right) (3 \text{ gal/min}), \quad x(0) = 30
\]

since there is a net gain of 2 gal of water into the tank every minute. This equation is separable: for \( x \neq 0 \),

\[
\int \frac{dx}{x} = -3 \int \frac{dt}{20 + 2t} \implies \ln |x| = -3 \left( \frac{1}{2} \right) \ln |20 + 2t| + C, \quad C \in \mathbb{R}
\]

\[
\implies |x| = \left( e^{\ln(20+2t)} \right)^{-3/2} \cdot e^C
\]

\[
\implies x(t) = \frac{A}{(20 + 2t)^{3/2}}, \quad A = e^C > 0
\]

where \( t \) and \( x(t) \) are certainly nonnegative so the absolute values are not necessary.
Using $x(0) = 30$ we solve for $A$:

$$30 = \frac{A}{(20 + 0)^{3/2}} \implies A = 30 \cdot 20^{3/2}.$$ 

(Note again: IVP solution is not necessary to receive full credit for this part; it is provided for reference).