1. (20 points) As part of an engineering art project, you are attempting to balance a plastic sheet on top of a needle. You know that in order to balance this plastic sheet, the center of mass of the sheet must be located on the sheet itself. The sheet lives in the upper half plane and is bounded by \( y = -x, \ y = x, \ x^2 + y^2 = 1, \) and \( x^2 + y^2 = 4, \) and has density \( \delta(x,y) = \frac{1}{\sqrt{x^2+y^2}}. \)

(a) Find the mass of the sheet.

(b) Find the center of mass of the sheet.

Solution:

\( m = \int_R \delta(x,y) \, dA = \int_0^{\pi/4} \int_1^2 \frac{1}{r} \, r \, dr \, d\theta = \frac{\pi}{2} \)

(b)

\( \bar{x} = \frac{1}{m} \int_R x \delta(x,y) \, dA = \frac{2}{\pi} \int_0^{\pi/4} \int_1^2 \cos \theta \, \frac{1}{r} \, r \, rd\theta = 0 \) (or just use symmetry to get 0)

\( \bar{y} = \frac{1}{m} \int_R y \delta(x,y) \, dA = \frac{2}{\pi} \int_0^{\pi/4} \int_1^2 \sin \theta \, \frac{1}{r} \, r \, rd\theta = \frac{2}{\pi} \sqrt{2} \sqrt{2} = \frac{3\sqrt{2}}{2} \)

2. (20 points) The level of spiciness of this exam can be calculated by finding the volume of the solid, \( E, \) that is bounded by:

\( x = -1 \quad y = -1 \quad z = 0 \)

\( x = 1 \quad y = 1 \quad z = x^2 - y^2 + 1 \)

(a) Using cartesian coordinates with the order \( dz \, dy \, dx, \) set up the triple integral that if evaluated would give the volume of \( E. \)

(b) Rewrite your integral using cartesian coordinates with the order \( dy \, dz \, dx. \)

Solution:

(a) \( V = \int_{-1}^1 \int_{-1}^1 \int_0^{x^2-y^2+1} \, dz \, dy \, dx \)

(b) We calculate the volume in first octant then multiply by 4,

\( V = 4 \int_0^1 \left\{ \int_0^{x^2+1} \int_0^{\sqrt{x^2+1-z}} \, dy \, dz + \int_0^{x^2} \int_0^1 \, dy \, dz \right\} \, dx \)

3. (30 points) Consider the following integral

\( I = \int_{x=0}^1 \int_{y=0}^{1-x} \sqrt{x+y(y-2x)^2} \, dy \, dx \)

(a) Sketch the region of integration in the \( xy \)-plane. Be sure to label all axes, boundaries, etc. on your sketch.

(b) Now use the transformation \( u = x+y, \ v = y-2x \) to transform the region into the corresponding region in the \( uv \) plane and make a sketch of it. Be sure to label all axes, boundaries, etc. on your sketch.

(c) Convert the original integral into one in terms of \( u \) and \( v. \) Select and order of integration that will result in one integral. Be sure to include the limits of integration.

(d) Evaluate one of the integrals to determine the value of \( I. \)
Solution:

(a)

(b) \( S = \{ (u, v) | 0 \leq u \leq 1, -2u \leq v \leq u \} \)

(c) Since

\[
\begin{align*}
x &= \frac{u - v}{3}, \\
y &= \frac{2u + v}{3},
\end{align*}
\]

Jacobian is

\[
\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \begin{array}{cc} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{array} \right| = \frac{1}{3}
\]

So,

\[
I = \iint_S \sqrt{uv^2} \frac{1}{3} \, dvdu = \int_0^1 \int_{-2u}^u \sqrt{uv^2} \frac{1}{3} \, dvdu
\]

(d)

\[
I = \frac{1}{9} \int_0^1 \sqrt{uv^3} \big|_{-2u}^u \, du = \int_0^1 u^2 \, du = \frac{2}{9}
\]

4. (30 points) The volume of an object is given by

\[
V = \int_{\theta=0}^{2\pi} \int_{r=0}^r \int_{z=0}^r r \, dz \, dr \, d\theta + \int_{\theta=0}^{2\pi} \int_{r=1}^{\sqrt{2}} \int_{z=0}^1 r \, dz \, dr \, d\theta
\]

(a) Plot the cross-section of the object in the \( rz \)-plane (this is a constant \( \theta \) plane in cylindrical coordinates) clearly labeling the boundaries of the object. (If you have trouble with this, you may "buy" a sketch of the shape of the region in the \( rz \)-plane with no labels for 5 points. This sketch will only show the shape of the region, so you will still need to supply the remaining details. The offer to buy this sketch ends after 75 minutes into the exam!)

(b) Rewrite the integral for \( V \) in cylindrical coordinates using the order \( dr \, dz \, d\theta \)

(c) Convert the integral for \( V \) in spherical coordinates using the order \( d\phi \, d\rho \, d\theta \).

(d) Rewrite the integral for \( V \) in spherical coordinates using the order \( d\rho \, d\phi \, d\theta \).

Solution:
(b) \[ V = \int_0^{2\pi} \int_0^1 \int_z^{\sqrt{2}} r \, dr \, dz \, d\theta. \]

(c) \[ V = \int_0^{2\pi} \left\{ \int_0^{\sqrt{2}} \int_{\pi/4}^{\pi/2} \rho^2 \sin \phi \, d\phi \, d\rho + \int_0^{\sqrt{2}} \int_{\cos^{-1} \frac{\sqrt{2}}{\rho}}^{\sin^{-1} \frac{\sqrt{2}}{\rho}} \rho^2 \sin \phi \, d\phi \, d\rho \right\} \, d\theta \]

(d) \[ V = \int_0^{2\pi} \left\{ \int_{\pi/4}^{\cos^{-1} \frac{1}{\sqrt{3}}} \int_0^{\cos^{-1} \frac{1}{\sqrt{3}}} \rho^2 \sin \phi \, d\rho \, d\phi + \int_{\cos^{-1} \frac{1}{\sqrt{3}}}^{\pi/2} \int_0^{\cos^{-1} \frac{1}{\sqrt{3}}} \rho^2 \sin \phi \, d\rho \, d\phi \right\} \, d\theta \]