1. (20 points) For each of the following unrelated questions, answer either ALWAYS TRUE or NOT ALWAYS TRUE. For this problem, no justification is necessary.

(a) If $\nabla f(a, b)$ and $\nabla g(a, b)$ point in opposite directions, then $(a, b)$ is the location of a local minimum of $f(x, y)$ constrained by $g(x, y) = k$.
(b) If $f(x, y)$ is a differentiable function, then there is a direction in which the rate of change of $f$ at $(a, b)$ is 0.
(c) If $f(x, y)$ approaches 1 as $(x, y)$ approaches $(0, 0)$ along the line $y = mx$, for every $m$, then $\lim_{(x, y) \to (0, 0)} f(x, y) = 1$.
(d) If $\nabla f(1, 2) = \hat{i}$, then $f(10, 2) > f(1, 2)$

Solution:

(a) NOT ALWAYS TRUE
(b) ALWAYS TRUE
(c) NOT ALWAYS TRUE
(d) NOT ALWAYS TRUE

2. (30 points) Sammi the shark is swimming along the path $\mathbf{r}(t)$ in the Nabla river looking for fish. The pressure distribution that morning in the river is $P(x, y, z)$. At some time $t^*$ (and only at this particular time), you know that $\mathbf{r}(t^*) = 3\hat{j} - \hat{k}$, $\mathbf{v}(t^*) = 2\hat{i} + 1\hat{j} - \hat{k}$, and $\mathbf{a}(t^*) = -1\hat{i} + 2\hat{j} + \hat{k}$. Furthermore, you know that $\nabla P|_{(0,3,-1)} = 2\hat{j} - 4\hat{k}$, and $P(0,3,-1) = 5$.

(a) As Sammi swims past location $\mathbf{r}(t^*)$, at what rate is the pressure, $P$, changing with respect to time, $dP/dt$?
(b) As she swims past location $\mathbf{r}(t^*)$ at what rate is the pressure $P$ changing with respect to distance traveled, $dP/ds$, where $s$ is arc length?
(c) If Sammi continues on her original path $\mathbf{r}(t)$ for a short interval of time $\Delta t = 0.1$, by approximately how much does the pressure change?
(d) On the other hand, suppose at time $t^*$ Sammi suddenly sees her favorite rock, and starts to swim towards it in a direction that happens to be the direction of the greatest rate of increase of $P$. Assuming Sammi maintains her same speed, by approximately how much does the pressure change after she flies for $\Delta t = 0.1$?

Solution:

(a) By the chain rule, we have:

$$\frac{dP}{dt} = \frac{d}{dt}P(x(t), y(t), z(t)) = \frac{\partial P}{\partial x}x'(t) + \frac{\partial P}{\partial y}y'(t) + \frac{\partial P}{\partial z}z'(t) = \nabla P|_{\mathbf{r}(t)} \cdot \mathbf{v}(t)$$

$$\Rightarrow \left. \frac{dP}{dt} \right|_{t^*} = \langle 0, 2, -4 \rangle \cdot \langle 2, 1, -1 \rangle = 6.$$
(b) Since we have already calculated $dP/dt$, we can use this to calculate $dP/ds$ with less work:

$$\frac{dP}{ds} = \frac{dP}{dt} \frac{dt}{ds}$$

$$\Rightarrow \frac{dP}{ds}\bigg|_{r(t^*)} = \frac{6}{\sqrt{6}} = \sqrt{6}.$$  

(c) We can approximate the change in pressure using differentials:

$$\Delta P \approx dP = \frac{\partial P}{\partial x} \frac{dx}{dt} dt + \frac{\partial P}{\partial y} \frac{dy}{dt} dt + \frac{\partial P}{\partial z} \frac{dz}{dt} dt$$

$$= \frac{dP}{dt} dt$$

$$= 6(0.1)$$

$$= 0.6$$

(d) Now, Sammi is swimming in the direction of the gradient (as this gives the direction of greatest increase in the pressure), so we have:

$$\Delta P \approx dP = \frac{dP}{ds} \frac{ds}{dt} dt$$

$$= \nabla P \cdot \nabla P |\nabla P| |\nabla P|$$

$$= |\nabla P| |\nabla P|$$

$$= \sqrt{20} \sqrt{6} \frac{1}{10}$$

$$= \frac{\sqrt{30}}{5}$$

3. (30 points) Dorian the Dragonslayer is on the hunt for a ravenous dragon. He does not know exactly where the dragon’s lair is, but based on recent sightings, Dorian has found that the likelihood of the lair being at a location is given by

$$f(x, y) = -\frac{1}{3}x^3 + 2x^2 - 3x - y^2 + 13$$

(a) Find all of the critical points of the likelihood function, $f$, and classify them.

(b) If the edge of the forest is described by $(x - 1)^2 + y^2 = 9$. Where is(are) the most likely location(s) of the lair in the forest?

Solution:

(a) For the critical points of $f$, we need to find where the gradient of $f$ is equal to 0.

$$f_x(x, y) = -x^2 + 4x - 3$$

$$0 \overset{set}{=} -x^2 + 4x - 3$$

$$\Rightarrow 0 = -(x - 3)(x - 1)$$

$$\Rightarrow x = 1 \text{ or } 3$$

$$f_y(x, y) = -2y$$

$$0 \overset{set}{=} -2y$$

$$\Rightarrow y = 0$$
So our critical points are (1, 0) and (3, 0). In order to classify these, we need to look at the discriminant

\[ D = f_{xx}f_{yy} + (f_{xy})^2 \]
\[ = (-2x + 4)(-2) + 0^2 \]
\[ = 4x - 8 \]

So for the critical point (1, 0), we have \( D < 0 \), which means there is a saddle at (1, 0). For (3, 0), we have \( D > 0 \) and \( f_{yy} < 0 \), which means that there is a local maximum at (3, 0) with a value of \( f(3, 0) = -9 + 2 \cdot 9 - 9 - 0 + 13 = 13 \).

(b) If we want the most likely location of the lair, we need to find the absolute maximum of the likelihood function, which means finding any maximums along the boundary of the forest. This is a Lagrange multiplier problem.

\[ f(x, y) = -\frac{1}{3}x^3 + 2x^2 - 3x - y^2 + 13 \]
\[ f_x = -x^2 + 4x - 3 \]
\[ f_y = -2y \]
\[ g(x, y) = (x - 1)^2 + y^2 = 9 \]
\[ g_x = 2(x - 1) \]
\[ g_y = 2y \]
\[-x^2 + 4x - 3 \overset{\text{set}}{=} \lambda 2(x - 1) \]
\[ \implies -(x - 1)(x - 3) = \lambda 2(x - 1) \]
\[-2y \overset{\text{set}}{=} \lambda 2y \]

From this system, we see that
\[ x = 1 \quad \text{or} \quad x = 3 - 2\lambda \]
\[ y = 0 \quad \text{or} \quad \lambda = -1 \]

\[ x = 1 \]
\[ (1 - 1)^2 + y^2 = 9 \]
\[ \implies y^2 = 9 \]
\[ \implies y = \pm 3 \]

\[ y = 0 \]
\[ (x - 1)^2 + 0^2 = 9 \]
\[ \implies (x - 1)^2 = 9 \]
\[ \implies x - 1 = \pm 3 \]
\[ \implies x = -2, 4 \]

\[ \lambda = -1 \]

\[ x = 3 + 2\lambda \]
\[ = 5 \]
\[ > 4 \ (\text{the bound on } x) \]

So the points of interest and their function values are:

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(f(x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, −3)</td>
<td>8/3</td>
</tr>
<tr>
<td>(1, 3)</td>
<td>8/3</td>
</tr>
<tr>
<td>(−2, 0)</td>
<td>89/3</td>
</tr>
<tr>
<td>(4, 0)</td>
<td>35/3</td>
</tr>
</tbody>
</table>

So the absolute maximum occurs at (−2, 0) with a likelihood of \(\frac{89}{3}\).
4. (20 points) Consider the function $f(x, y) = \sqrt{xy}$

(a) Find the first order Taylor series approximation for $f(x, y)$ about $\left(\frac{1}{2}, \frac{1}{2}\right)$.

(b) Find an upper bound on the error of the Taylor series approximation if $|x - \frac{1}{2}| \leq \frac{1}{4}$ and $|y - \frac{1}{2}| \leq \frac{1}{4}$.

Hint: Recognizing that for $x > 0$ and $y > 0$ we have $\sqrt{xy} = \sqrt{x} \sqrt{y}$ may make your derivatives look simpler.

(c) Use your approximation from part (a) to approximate $\sqrt{\frac{3}{8}}$. Hint: $\frac{3}{8} = \frac{1}{2} \frac{3}{4}$.

Solution:

(a) We know that $T_1(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$, so

$$a = \frac{1}{2}$$

$$b = \frac{1}{2}$$

$$f(a, b) = \sqrt{\left(\frac{1}{2}\right) \left(\frac{1}{2}\right)} = \frac{1}{4}$$

$$f_x(x, y) = \frac{\sqrt{y}}{2 \sqrt{x}}$$

$$f_x(a, b) = \frac{1/2}{2(1/2)} = 1/2$$

$$f_y(x, y) = \frac{\sqrt{x}}{2 \sqrt{y}}$$

$$f_y(a, b) = \frac{1/2}{2(1/2)} = 1/2$$

$$T_1(x, y) = \frac{1}{2} + \frac{1}{2} \left(x - \frac{1}{2}\right) + \frac{1}{2} \left(y - \frac{1}{2}\right)$$

$$= \frac{1}{2} x + \frac{1}{2} y$$

(b) If we want to bound this error, we need to use the fact that

$$|\text{error}| \leq \frac{M}{2} (|x - a| + |y - b|)^2$$

So, we have

$$f_{xx}(x, y) = -\frac{\sqrt{y}}{4 \sqrt{x}^3}$$

$$|f_{xx}| \leq \frac{\sqrt{3/4}}{4 \sqrt{1/4}}$$

$$= \sqrt{3}$$

$$f_{xy}(x, y) = \frac{1}{4 \sqrt{x} \sqrt{y}}$$

$$|f_{xy}| \leq \frac{1}{4 \sqrt{1/4} \sqrt{1/4}}$$

$$= 1$$
\[ f_{yy}(x, y) = -\frac{\sqrt{x}}{4\sqrt{y}^3} \]

\[ |f_{yy}| \leq \frac{\sqrt{3/4}}{4\sqrt{1/4}^3} \]

\[ \implies M = \sqrt{3} \]

\[ \implies |\text{error}| \leq \frac{\sqrt{3}}{2} \left( \frac{1}{4} + \frac{1}{4} \right)^2 \]

\[ = \frac{\sqrt{3}}{2} \cdot \frac{1}{4} \]

\[ = \frac{\sqrt{3}}{8} \]

(c) Since \( \frac{3}{8} = \frac{13}{24} \), we have

\[ \sqrt{\frac{3}{8}} = \sqrt{\frac{13}{24}} \approx T_1(1/2, 3/4) = \frac{5}{8} \]