INSTRUCTIONS: Electronic devices are not permitted during the exam. Write your name, your instructor’s name, and your recitation number on the front of your bluebook. Start each problem on a new right-hand page. Be sure to clearly justify your work. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (25 points) Alvin the ant is out looking for food when, for no particular reason, he decides to do the “Harlem Shake.” As he dances for \(t \geq 0\), he moves along the path

\[
\mathbf{r}(t) = (1 - \cos(t))\mathbf{i} + (t - \sin(t))\mathbf{j} + 0\mathbf{k}.
\]

(a) Determine Alvin’s velocity, \(\mathbf{v}(t)\), and his acceleration, \(\mathbf{a}(t)\).

(b) Is Alvin’s acceleration \(\mathbf{a}(t)\) ever orthogonal to his velocity \(\mathbf{v}(t)\)? If so, when?

(c) If Alvin dances for \(0 \leq t \leq 2\pi\), how far has he traveled?

(d) If Alvin dances for \(0 \leq t \leq 4\pi\), how far has he traveled?

2. (25 points) As a last line of defense against attacks, the missile space station “Seb’s Hammer” orbits around the planet Skartron–Pax (which is located at the origin) along the trajectory

\[
\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sqrt{3}\sin(\pi t)\mathbf{j} + \sqrt{2}\cos(\pi t)\mathbf{k},
\]

where \(t\) is in units of hours and \(x, y, z\) are in units of juno-meters.

(a) Calculate \(\mathbf{T}\) and \(\mathbf{N}\) (both as a function of time, if appropriate) for the trajectory of “Seb’s Hammer.”

(b) Calculate the speed and curvature (both as a function of time, if appropriate) for “Seb’s Hammer” along its path.

(c) What are the tangential and normal components, \(a_T\) and \(a_N\), of the acceleration of “Seb’s Hammer?” Hint: recall that \(\mathbf{a}(t) = a_T\mathbf{T} + a_N\mathbf{N}\).

(d) At time \(t = 3\) hours, “Seb’s Hammer” detects an enemy craft from the evil planet Norcon–Ω straight ahead at a distance of 4 juno-meters away. At what point \((x, y, z)\) is the enemy craft located?

3. (25 points) A triangular flag hangs on a flagpole whose base is located at \((0,0,0)\). The flag is attached to the flagpole at \((0,0,2)\) and \((0,0,4)\). A strong wind blowing in (approximately) the Northeast direction causes the flag to stretch out fully such that the third corner of the flag is located at \((1,2,3)\).

(a) Determine the equation of the plane containing the flag.

(b) Calculate the shortest distance from the third corner to the flagpole. (For full credit here, be sure to use Calculus III concepts here, not just simple High School geometry.)

(c) The winds now change so that the flag is fully stretched out but pointing in (approximately) the Southeast direction at an angle that is perpendicular to its previous direction. What is the equation for the new plane that contains the flag? (Again, for full credit here, be sure to use Calculus III concepts here, not just simple High School geometry.)

OVER
4. (25 points) Consider the two quadric surfaces given by \( z = 2 - x^2 - y^2 \) and \( z = y^2 - ax^2 \), where \( a \) is a constant. Where these surfaces intersect, they form a path in 3-D space. The shadow of the intersection path onto the \( x-y \) plane will form “shadow curves” of various shapes. Your job is to find the parameterization \( \mathbf{r}(t) = x(t) \hat{i} + y(t) \hat{j} \) that represents the “shadow curves” for the cases described below.

(a) \( a = 1 \) (There are two shadow curves here, so be sure to parameterize both!)

(b) \( a = \frac{1}{2} \)

(c) \( a = 3 \) (Again, there are two shadow curves here, so be sure to parameterize both! Hint: one possible parameterization for the hyperbola \( \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \) is \( \mathbf{r}(t) = a \tan(t) \hat{i} \pm b \sec(t) \hat{j} \).)

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Projections and distances from a point to a line and a plane

\[
\proj_{\mathbf{A}} \mathbf{B} = \left( \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \right) \mathbf{A}
\]

\[
d = \frac{\left| \vec{PS} \times \mathbf{v} \right|}{\left| \mathbf{v} \right|}
\]

\[
d = \frac{\left| \vec{PS} \cdot \mathbf{n} \right|}{\left| \mathbf{n} \right|}
\]

Arc length, Frenet formulas, and tangential and normal acceleration components

\[
ds = \left| \mathbf{v} \right| dt
\]

\[
\hat{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{\left| \mathbf{v} \right|}
\]

\[
\hat{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left| \frac{d\mathbf{T}}{ds} \right|} = \frac{\frac{d\mathbf{T}}{dt}}{\left| \frac{d\mathbf{T}}{dt} \right|}
\]

\[
\hat{B} = \hat{T} \times \hat{N}
\]

\[
\frac{d\hat{T}}{ds} = \kappa \hat{N}
\]

\[
\frac{d\hat{B}}{ds} = -\tau \hat{N}
\]

\[
\kappa = \left| \frac{\mathbf{v} \times \mathbf{a}}{\left| \mathbf{v} \right|^3} \right|
\]

\[
\tau = \frac{\left| \mathbf{v} \times \mathbf{a} \right|}{\left| \mathbf{v} \right|}
\]

\[
a = a_N \hat{N} + a_T \hat{T}
\]

\[
a_T = \frac{d\left| \mathbf{v} \right|}{dt}
\]

\[
a_N = \kappa \left| \mathbf{v} \right|^2 = \sqrt{\left| \mathbf{a} \right|^2 - a_T^2}
\]

Fond memories of Calculus I and II

\[
\cos^2 \theta = \frac{1 + \cos 2\theta}{2}
\]

\[
\sin^2 \theta = \frac{1 - \cos 2\theta}{2}
\]

\[
\sin^2 \theta + \cos^2 \theta = 1
\]

\[
\tan^2 \theta + 1 = \sec^2 \theta
\]

\[
1 + \cot^2 \theta = \csc^2 \theta
\]