1. (20pts) For each part answer **ALWAYS TRUE** or **NOT ALWAYS TRUE**. No justification is required.

(a) If $u$ is a unit vector and $f(x, y) = \sin x + \cos y$ then $-\sqrt{2} \leq D_u f \leq \sqrt{2}$ for all points $(x, y)$.

(b) If $g(t)$ is defined for all $t$ then the curve $r(t) = \sin t \ i + \cos t \ j + g(t) \ k$ lies on the cylinder $x^2 + y^2 = 1$.

(c) If the circulation of a vector field $F$ around some closed path $r(t)$ is zero then $F$ is conservative.

(d) If $S$ is a closed oriented surface and $F$ is a constant vector field then $\int_S F \cdot n \ d\sigma = 0$.

2. (30 pts) A space station (not a moon) orbits a planet on the path described by $5x^2 + 6xy + 5y^2 = 1$. Use Calculus III techniques to find the points where the space station is closest to and farthest from the planet if the planet is located at the point $(0, 0)$.

(a) Calculate the rate of change of efficiency, $E$, with respect to time, along the path $r(t)$. You may leave your answer in terms of functions of $t$ and partial derivatives of $c(x, y, z)$ with respect to $x$, $y$, and $z$.

(b) Estimate the change in efficiency, $\Delta E$, if the pod follows its current course at time $t = \pi$ for $\Delta t = 0.2$ seconds if you know that at this time (and only this time) $\nabla c = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

(c) Calculate the rate of change of efficiency, $E$, with respect to distance (arc length), along the path $r(t)$. You may leave your answer in terms of functions of $t$ and partial derivatives of $c(x, y, z)$ with respect to $x$, $y$, and $z$.

(d) Estimate the change in efficiency, $\Delta E$, if the pod follows its current course at time $t = \pi$ for $\Delta s = 0.1$ meters if you know that at this time (and only this time) $\nabla c = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

4. (30 pts) Consider the force field $\mathbf{F} = b \sin y \ \mathbf{i} + (x \cos y + b \cos z) \ \mathbf{j} + c y \sin z \ \mathbf{k}$ acting on a particle moving on the path $r(t) = t \sin^3 t \ \mathbf{i} + t \cos^3 t \ \mathbf{j} + t \ \mathbf{k}$ for $0 \leq t \leq 2\pi$.

(a) For what values of $b$ and $c$ will $\mathbf{F}$ be conservative? Be sure to justify your conclusion. You might want to check this twice because the rest of the problem depends on it.

(b) Using your values of $b$ and $c$, determine a potential function for $\mathbf{F}$.

(c) Using your values of $b$ and $c$, determine the work done by $\mathbf{F}$ on the particle.

5. (40 pts) Consider the upward oriented open surface $S$ defined by $z = 4 - x^2 - y^2$ for $z \geq 0$ and the field $\mathbf{F} = (x + y) \ \mathbf{i} + (y - x) \ \mathbf{j} + (z - 1) \ \mathbf{k}$.

(a) Determine the flux of $\mathbf{F}$ through $S$ by direct computation.

(b) Verify your result from part (a) using an appropriate theorem from Calculus III.

6. (40 pts) Consider the surface $S$ defined by the part of the cone $z^2 = x^2 + y^2$ in the first octant bounded between the planes $z = 1$ and $z = 2$ and let $C$ be the boundary curve of $S$ oriented counterclockwise when viewed from above. Finally, consider the field $\mathbf{F} = y \ (1 + z^2) \ \mathbf{i} + x \ \mathbf{j} + xyz \ \mathbf{k}$.

(a) Make a clear sketch of $S$ and its boundary curve.

(b) Parameterize the boundary curve $C$. Be sure to clearly state the range of $t$-values for each part of the curve.

(c) Determine the circulation of $\mathbf{F}$ around $C$ by direct computation.

(d) Verify your result from part (a) using an appropriate theorem from Calculus III.

ENJOY YOUR BREAK!
Projections and distances \( \text{proj}_A B = \left( \frac{A \cdot B}{A \cdot A} \right) A \quad d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = d = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{|\mathbf{n}|} \)

Arc length, frenet formulas, and tangential and normal acceleration components

\[
ds = |\mathbf{v}| dt \quad \mathbf{T} = \frac{dr}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \mathbf{N} = \frac{dT}{|dT|} = \frac{dT}{dt} \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}
\]

\[
\frac{dT}{ds} = \kappa \mathbf{N} \quad \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N} \quad \kappa = \frac{dT}{ds} = \frac{\mathbf{v} \times \alpha}{|\mathbf{v}|^3} = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|\ddot{y} - \ddot{z}|}{|\ddot{x} + \ddot{y}|^{3/2}} \quad \tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}
\]

\[
a = aT \mathbf{N} + a\tau \mathbf{T} \quad aT = \frac{d|\mathbf{v}|}{dt} \quad aN = \kappa |\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}
\]

The Second Derivative Test
Suppose \( f(x, y) \) and its first and second partial derivatives are continuous in a disk centered at \((a, b)\), and \( f_x(a, b) = f_y(a, b) = 0 \). Let \( D = f_{xx} f_{yy} - f_{xy}^2 \).

1. If \( D > 0 \) and \( f_{xx} < 0 \) at \((a, b)\), then \( f \) has a local maximum at \((a, b)\).
2. If \( D > 0 \) and \( f_{xx} > 0 \) at \((a, b)\), then \( f \) has a local minimum at \((a, b)\).
3. If \( D < 0 \) at \((a, b)\), then \( f \) has a saddle point at \((a, b)\).
4. If \( D = 0 \) at \((a, b)\), then the test is inconclusive.

Directional derivative, discriminant, and Lagrange multipliers

\[
\frac{df}{ds} = D_u f = (\nabla f) \cdot \mathbf{u} \quad f_{xx} f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda \nabla g, \; g = 0
\]

Taylor’s formula (at the point \((x_0, y_0)\))

\[
f(x, y) = f(x_0, y_0) + \left[ (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) \right] + \frac{1}{2} \left[ (x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0) f_{xy}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0) \right] + \frac{1}{3} \left[ (x - x_0)^3 f_{xxx}(x_0, y_0) + (y - y_0)^2 f_{xyy}(x_0, y_0) + 3(x - x_0)(y - y_0)^2 f_{xxy}(x_0, y_0) + (y - y_0)^3 f_{yyy}(x_0, y_0) \right] + \cdots
\]

Linear approximation error

\[
|E(x, y)| \leq \frac{M}{2!} (|x - x_0| + |y - y_0|)^2 , \quad \text{where max} \left\{|f_{xx}|, |f_{xy}|, |f_{yy}| \right\} \leq M
\]

Polar coordinates

\[
x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad dA = dx \, dy = r \, dr \, d\theta
\]

Cylindrical and spherical coordinates

<table>
<thead>
<tr>
<th>Cylindrical to Rectangular</th>
<th>Spherical to Cylindrical</th>
<th>Spherical to Rectangular</th>
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<tbody>
<tr>
<td>( x = r \cos \theta )</td>
<td>( r = \rho \sin \phi )</td>
<td>( x = \rho \sin \phi \cos \theta )</td>
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<tr>
<td>( y = r \sin \theta )</td>
<td>( z = \rho \cos \phi )</td>
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<td>( z = z )</td>
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<td>( z = \rho \cos \phi )</td>
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\[
dV = dx \, dy \, dz = dz \, r \, d\theta \, dr = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

Substitutions in multiple integrals

\[
\iint_R f(x, y) \, dx \, dy = \iiint_G f(x(u, v), y(u, v)) \left| J(u, v) \right| \, du \, dv \quad \text{where} \quad J(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \left| \begin{array}{ll} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right|
\]

Mass, moments, and center of mass

\[
\text{Mass} \quad M = \iint_R \delta \, dA
\]

Moments \( M_x = \iint_R y \, \delta \, dA \) \( M_y = \iint_R x \, \delta \, dA \) Center of mass \( \bar{x} = M_y / M \) \( \bar{y} = M_x / M \)

Green’s Theorem in the \( x-y \) plane

The curve \( C \) is traversed counterclockwise, and \( \mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} \).

Circulation \( = \oint_C \mathbf{F} \cdot T \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA = \int_R \mathbf{Q_x - P_y} \, dA \)

Outward Flux \( = \oint_C \mathbf{F} \cdot n \, ds = \int_C \mathbf{F} \cdot n \, dS = \int R \mathbf{P_x + Q_y} \, dA \)

Surface area of level surface \( g(x, y, z) = c \)

\[SA = \iint_S \delta \, ds = \int_R \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} \, dA\]

Stokes’ Thm: \( \oint_C \mathbf{F} \cdot T \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot n \, dS \quad \text{Divergence Thm:} \quad \int_S \mathbf{F} \cdot n \, dS = \iiint_E \nabla \cdot \mathbf{F} \, dV \)

Fond memories

\[
\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}
\]
Solutions

1. (a) **always true**
   (b) **always true**
   (c) **not always true**
   (d) **always true**

2. We want to minimize/maximize the squared distance from the origin subject to the constraint that the point \((x, y)\) lies on the given path. We then have

\[
\begin{align*}
\text{min}/\text{max} \quad D(x, y) &= x^2 + y^2 \\
\text{subject to} \quad g(x, y) &= 5x^2 + 6xy + 5y^2 = 1 \\
\partial D &= \lambda \partial g \\
g &= 1
\end{align*}
\]

This gives the system of equations

\[
\begin{align*}
2x &= \lambda (10x + 6y) \\
2y &= \lambda (10y + 6x) \\
5x^2 + 6xy + 5y^2 &= 1
\end{align*}
\]

Multiplying (1) by \(y\) and (2) by \(x\) and setting the resulting right-hand sides equal, we have

\[
6y^2 = 6x^2
\]

which gives \(y = \pm x\). Plugging \(y = x\) into the constraint gives

\[
5x^2 + 6x^2 + 5x^2 = 1 \quad \Rightarrow \quad 16x^2 = 1 \quad \Rightarrow \quad x = \pm \frac{1}{4} \quad \Rightarrow \quad \left( \frac{1}{4}, \frac{1}{4} \right), \left( -\frac{1}{4}, -\frac{1}{4} \right)
\]

Plugging in \(y = -x\) into the constraint gives

\[
5x^2 - 6x^2 + 5x^2 = 1 \quad \Rightarrow \quad 4x^2 = 1 \quad \Rightarrow \quad x = \pm \frac{1}{2} \quad \Rightarrow \quad \left( \frac{1}{2}, -\frac{1}{2} \right), \left( -\frac{1}{2}, \frac{1}{2} \right)
\]

We then have

\[
D \left( \pm \frac{1}{4}, \pm \frac{1}{4} \right) = \frac{1}{8} \quad D \left( \pm \frac{1}{2}, \pm \frac{1}{2} \right) = \frac{1}{2}
\]

So the space station is at its maximum distance from the planet at the points \((\pm 1/2, \mp 1/2)\) and its minimum distance at the points \((\pm 1/4, \pm 1/4)\).
3. (a) We have

\[ \mathbf{r}(t) = (1 - \sin t) \mathbf{i} + 2 \cos t \mathbf{j} + \left(1 - \sqrt{3} \sin t\right) \mathbf{k} \]

\[ \frac{dE}{dt} = 3 \frac{dz}{dt} - \frac{dc}{dt} = 3 \frac{dz}{dt} - \frac{\partial c}{\partial x} \frac{dx}{dt} - \frac{\partial c}{\partial y} \frac{dy}{dt} - \frac{\partial c}{\partial z} \frac{dz}{dt} = c_x \cos t + 2c_y \sin t + \sqrt{3} (c_z - 3) \cos t \]

(b) We have

\[ \frac{dE}{dt} = 2 \quad \Rightarrow \quad \Delta E \approx \frac{dE}{dt} \Delta t = 0.4 \]

(c) We have

\[ \frac{dE}{ds} = \frac{dE}{dt} \frac{dt}{ds} = \frac{dE}{dt} \frac{1}{|\mathbf{v}|}, \quad \mathbf{v} = -\cos t \mathbf{i} - 2 \sin t \mathbf{j} - \sqrt{3} \cos t \mathbf{k}, \quad |\mathbf{v}| = 2 \]

Then

\[ \frac{dE}{ds} = \frac{1}{2} \left( c_x \cos t + 2c_y \sin t + \sqrt{3} (c_z - 3) \cos t \right) \]

(d) We have

\[ \frac{dE}{ds} = 1 \quad \Rightarrow \quad \Delta E \approx \frac{dE}{ds} \Delta s = 0.1 \]
4. (a) We want to determine values of $b$ and $c$ such that $\nabla \times \mathbf{F} = \mathbf{0}$. We have

$$
\nabla \times \mathbf{F} = \begin{vmatrix}
i & j & k \\
\partial_x & \partial_y & \partial_z \\
b \sin y & x \cos y + b \cos z & c y \sin z
\end{vmatrix} = (c \sin z + b \sin z, 0, \cos y - b \cos y)
$$

From the $k$-component of the curl we have $b = 1$. Then from the $i$ component of the curl we have $c = -1$. So the field becomes

$$
\mathbf{F} = \sin y \mathbf{i} + (x \cos y + \cos z) \mathbf{j} - y \sin z \mathbf{k}
$$

(b) We have $\mathbf{F} = \langle f_x, f_y, f_z \rangle$. Then

$$
\begin{align*}
f_x &= \sin y \Rightarrow f = x \sin y + g(y, z) \\
f_y &= x \cos y + \cos z \Rightarrow f = x \sin y + y \cos z + h(x, z) \\
f_z &= -y \sin z \Rightarrow f = y \cos z + k(x, y)
\end{align*}
$$

So the potential function is $f = x \sin y + y \cos z$.

(c) The path starts on the point $A = (0, 0, 0)$ and ends at the point $B = (0, 2\pi, 2\pi)$. Then we have by the Fundamental Theorem of Line Integrals:

$$
W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 2\pi, 2\pi) - f(0, 0, 0) = (0 + 2\pi \cos(2\pi)) - (0 + 0 \cos 0) = 2\pi
$$
5. (a) By direct calculation, we have

\[ \text{flux}_{\text{top}} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R \mathbf{F} \cdot \frac{\pm \nabla g}{|\nabla g \cdot \mathbf{p}|} \, dA \]

We’ll choose to project the surface into the \( xy \)-plane, so we have \( \mathbf{p} = \mathbf{k} \). Then

\[ g = z + x^2 + y^2 \quad \nabla g = \langle 2x, 2y, 1 \rangle \quad |\nabla g \cdot \mathbf{k}| = 1 \]

Then

\[ \mathbf{F} \cdot \mathbf{n} \, d\sigma = \langle x + y - x, y - x, -1 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dA = (2x^2 + 2y^2 + z - 1) \, dA = (x^2 + y^2 + 3) \, dA \]

Integrating in polar coordinates, we have

\[ \text{flux}_{\text{top}} = \int_0^{2\pi} \int_0^2 r^3 + 3r \, dr \, d\theta = \int_0^{2\pi} \frac{1}{4} r^4 + \frac{3}{2} r^2 \bigg|_0^2 \, d\theta = 20\pi \]

(b) An alternative method is to cap the surface by a disc in the \( xy \)-plane, compute the flux through the composite surface using the Divergence theorem, and then subtract off the flux through the bottom surface. We have

\[ \nabla \cdot \mathbf{F} = 3 \quad \text{flux}_{\text{total}} = \iiint_E 3 \, dV = 3 \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r \, dz \, dr \, d\theta \]

\[ = 3 \int_0^{2\pi} \int_0^2 4r - r^3 \, dr \, d\theta = 3 \int_0^{2\pi} 2r^2 - \frac{1}{4} r^4 \bigg|_0^2 \, d\theta = 12 \int_0^{2\pi} 2 \, d\theta = 24\pi \]

The flux through the disc in the \( xy \)-plane is given by

\[ \text{flux}_{\text{bottom}} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R (x + y - x, y - x, -1) \cdot (0, 0, -1) \, dA = \iint_R 0 \, dA = 4\pi \]

We then have

\[ \text{flux}_{\text{top}} = \text{flux}_{\text{total}} - \text{flux}_{\text{bottom}} = 20\pi \]
6. (a) The surface and curve look as follows

\[ \{ (2, 0, 2), (1, 0, 1), (0, 1, 1), (0, 2, 2) \} \]

(b) We have \( C = C_1 \cup C_2 \cup C_3 \cup C_4 \) where

\[
C_1 : \quad \mathbf{r}(t) = (2 \cos t, 2 \sin t, 2) \quad 0 \leq t \leq \pi/2 \\
C_2 : \quad \mathbf{r}(t) = (0, 2 - t, 2 - t) \quad 0 \leq t \leq 1 \\
C_3 : \quad \mathbf{r}(t) = (\sin t, \cos t, 1) \quad 0 \leq t \leq \pi/2 \\
C_4 : \quad \mathbf{r}(t) = (1 + t, 0, 1 + t) \quad 0 \leq t \leq 1
\]

(c) We have

\[
\begin{align*}
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} \left( 10 \sin t, 2 \cos t, 8 \sin t \cos t \right) \cdot \left( -2 \sin t, 2 \cos t, 0 \right) dt = \int_0^{\pi/2} 4 \cos^2 t - 20 \sin^2 t \, dt \\
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \left( (2 - t) \left[ 1 + (2 - t)^2 \right], 0, 0 \right) \cdot (0, -1, -1) \, dt = 0 \\
\int_{C_3} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} \left( 2 \cos t, \sin t, \sin t \cos t \right) \cdot \left( \cos t, -\sin t, 0 \right) dt = \int_0^{\pi/2} 2 \cos^2 t - \sin^2 t \, dt \\
\int_{C_4} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (0, 1 + t, 0) \cdot (1, 0, 1) \, dt = 0
\end{align*}
\]

Then

\[
\text{circ} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} 4 \cos^2 t - 20 \sin^2 t \, dt + \int_0^{\pi/2} 2 \cos^2 t - \sin^2 t \, dt = \int_0^{\pi/2} 6 \cos^2 t - 21 \sin^2 t \, dt
\]

\[
= \int_0^{\pi/2} \frac{6}{2} (1 + \cos (2t)) - \frac{21}{2} (1 - \cos (2t)) \, dt = \int_0^{\pi/2} -\frac{15}{2} + \frac{27}{2} \cos (2t) \, dt = -\frac{15\pi}{4}
\]

(d) Using Stokes’ Theorem with the cone as the surface, we have

\[
\text{circ} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \frac{\pm \nabla g}{|\nabla g \cdot \mathbf{p}|} \, dA
\]

We choose to project the cone down into the \( xy \)-plane, so we have \( \mathbf{p} = \mathbf{k} \). The region of integration is the area between the two quarter circles of radii 1 and 2. We have

\[
\nabla \times \mathbf{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\partial_x & \partial_y & \partial_z \\
y & x & yz
\end{vmatrix} = \langle xz, yz, -z^2 \rangle
\]

\[
g = z^2 - x^2 - y^2 = 0 \quad \nabla g = \langle -2x, -2y, 2z \rangle \quad |\nabla g \cdot \mathbf{k}| = |2z|
\]

\[
\nabla \times \mathbf{F} \cdot d\mathbf{r} = \langle xz, yz, -z^2 \rangle \cdot \frac{-x, -y, z}{z} \, dA = \langle -x^2 - y^2 - z^2 \rangle \, dA = -2 (x^2 + y^2) \, dA
\]

Integrating in polar coordinates, we have

\[
\text{circ} = \int_0^{\pi/2} \int_0^2 -2r^2 \sin \theta \, dr \, d\theta = \frac{1}{2} \int_0^{\pi/2} \frac{r}{2} \bigg|_0^2 \, d\theta = -\frac{15\pi}{4}
\]