1. (20pts) For each part answer **ALWAYS TRUE** or **NOT ALWAYS TRUE**. No justification is required.

(a) If \( \mathbf{u} \) is a unit vector and \( f(x, y) = \sin x + \cos y \) then \( -\sqrt{2} \leq D_u f \leq \sqrt{2} \) for all points \((x, y)\).

(b) If \( g(t) \) is defined for all \( t \) then the curve \( \mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + g(t) \mathbf{k} \) lies on the cylinder \( x^2 + y^2 = 1 \).

(c) If the circulation of a vector field \( \mathbf{F} \) around some closed path \( \mathbf{r}(t) \) is zero then \( \mathbf{F} \) is conservative.

(d) If \( S \) is a closed oriented surface and \( \mathbf{F} \) is a constant vector field then \( \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0 \).

2. (30 pts) A space station (not a moon) orbits a planet on the path described by \( 5x^2 + 6xy + 5y^2 = 1 \). Use Calculus III techniques to find the points where the space station is closest to and farthest from the planet if the planet is located at the point \((0,0)\).

3. (40 pts) Podracing is a popular sport on desolate, desert planets where racers perform numerous laps on dangerous courses while often reaching speeds of up to 700 km/hr. A racing craft, or **pod**, is typically composed of a small, one-man cockpit pulled along by two large engines. The efficiency of the engines, \( E \), is a function of the dust concentration in the air, \( c \), and the altitude, \( z \), as described by the function \( E(c, z) = 3z - c \). The local dust concentration is given by an unspecified function \( c(x, y, z) \). The path of this particular course is described by \( \mathbf{r}(t) = (1 - \sin t) \mathbf{i} + 2 \cos t \mathbf{j} + (1 - \sqrt{3} \sin t) \mathbf{k} \) for \( 0 \leq t \leq 2\pi \).

(a) Calculate the rate of change of efficiency, \( E \), with respect to time, along the path \( \mathbf{r}(t) \). You may leave your answer in terms of functions of \( t \) and partial derivatives of \( c(x, y, z) \) with respect to \( x \), \( y \), and \( z \).

(b) Estimate the change in efficiency, \( \Delta E \), if the pod follows its current course at time \( \Delta t = 0.2 \) seconds if you know that at this time (and only this time) \( \nabla c = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k} \).

(c) Calculate the rate of change of efficiency, \( E \), with respect to distance (arc length), along the path \( \mathbf{r}(t) \). You may leave your answer in terms of functions of \( t \) and partial derivatives of \( c(x, y, z) \) with respect to \( x \), \( y \), and \( z \).

(d) Estimate the change in efficiency, \( \Delta E \), if the pod follows its current course at time \( \Delta s = 0.1 \) meters if you know that at this time (and only this time) \( \nabla c = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k} \).

4. (30 pts) Consider the force field \( \mathbf{F} = b \sin y \mathbf{i} + (x \cos y + b \cos z) \mathbf{j} + c y \sin z \mathbf{k} \) acting on a particle moving on the path \( \mathbf{r}(t) = t \sin^3 t \mathbf{i} + t \cos^3 t \mathbf{j} + t \mathbf{k} \) for \( 0 \leq t \leq 2\pi \).

(a) For what values of \( b \) and \( c \) will \( \mathbf{F} \) be conservative? Be sure to justify your conclusion. You might want to check this twice because the rest of the problem depends on it.

(b) Using your values of \( b \) and \( c \), determine a potential function for \( \mathbf{F} \).

(c) Using your values of \( b \) and \( c \), determine the work done by \( \mathbf{F} \) on the particle.

5. (40 pts) Consider the upward oriented open surface \( S \) defined by \( z = 4 - x^2 - y^2 \) for \( z \geq 0 \) and the field \( \mathbf{F} = (x + y) \mathbf{i} + (y - x) \mathbf{j} + (z - 1) \mathbf{k} \).

(a) Determine the flux of \( \mathbf{F} \) through \( S \) by direct computation.

(b) Verify your result from part (a) using an appropriate theorem from Calculus III.

6. (40 pts) Consider the surface \( S \) defined by the part of the cone \( z^2 = x^2 + y^2 \) in the first octant bounded between the planes \( z = 1 \) and \( z = 2 \) and let \( C \) be the boundary curve of \( S \) oriented counterclockwise when viewed from above. Finally, consider the field \( \mathbf{F} = y(1 + z^2) \mathbf{i} + x \mathbf{j} + xyz \mathbf{k} \).

(a) Make a clear sketch of \( S \) and its boundary curve.

(b) Parameterize the boundary curve \( C \). Be sure to clearly state the range of \( t \)-values for each part of the curve.

(c) Determine the circulation of \( \mathbf{F} \) around \( C \) by direct computation.

(d) Verify your result from part (a) using an appropriate theorem from Calculus III.

**ENJOY YOUR BREAK!**
Projections and distances \[ \text{proj}_B A = \left( \frac{A \cdot B}{A \cdot A} \right) A \] \[ d = \frac{|\vec{P} \times \vec{v}|}{|\vec{v}|} \] \[ d = |\vec{P} \cdot \frac{\vec{n}}{|\vec{n}|}| \]

Arc length, frenet formulas, and tangential and normal acceleration components
\[ ds = |v| \, dt \quad \mathbf{T} = \frac{dr}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \mathbf{N} = \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt} \quad \mathbf{B} = \mathbf{T} \times \mathbf{N} \]
\[ \frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \] \[ \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N} \] \[ \kappa = \frac{|d\mathbf{T}|}{ds} = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{1+(f'(x))^2} \frac{1}{|\ddot{\mathbf{r}} + \mathbf{g}|} \] \[ \tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} \]

The Second Derivative Test
Suppose \( f(x, y) \) and its first and second partial derivatives are continuous in a disk centered at \((a, b)\), and \( f_x(a, b) = f_y(a, b) = 0 \). Let \( D = f_{xx}f_{yy} - f_{xy}^2 \).
1. If \( D > 0 \) and \( f_{xx} < 0 \) at \((a, b)\), then \( f \) has a local maximum at \((a, b)\).
2. If \( D > 0 \) and \( f_{xx} > 0 \) at \((a, b)\), then \( f \) has a local minimum at \((a, b)\).
3. If \( D < 0 \) at \((a, b)\), then \( f \) has a saddle point at \((a, b)\).
4. If \( D = 0 \) at \((a, b)\), then the test is inconclusive.

Directional derivative, discriminant, and Lagrange multipliers
\[ \frac{df}{ds} = D_u f = (\nabla f) \cdot \mathbf{u} \quad f_{xx}f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda \nabla g, \quad g = 0 \]

Taylor's formula (at the point \((x_0, y_0)\))
\[ f(x, y) = f(x_0, y_0) + (x-x_0)f_x(x_0, y_0) + (y-y_0)f_y(x_0, y_0) \]
\[ + \frac{1}{2!} \left[ (x-x_0)^2f_{xx}(x_0, y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0, y_0) + (y-y_0)^2f_{yy}(x_0, y_0) \right] \]
\[ + \frac{1}{3!} \left[ (x-x_0)^3f_{xxx}(x_0, y_0) + 3(x-x_0)^2(y-y_0)f_{xxy}(x_0, y_0) + 3(x-x_0)(y-y_0)^2f_{xyy}(x_0, y_0) + (y-y_0)^3f_{yyy}(x_0, y_0) \right] + \cdots \]

Linear approximation error
\[ |E(x, y)| \leq \frac{M}{2!} (|x-x_0| + |y-y_0|)^2 \]
where max \(|f_{xx}|, |f_{xy}|, |f_{yy}|\) \( \leq M \)

Polar coordinates
\[ x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad dA = dx \, dy = r \, dr \, d\theta \]

Cylindrical and spherical coordinates

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<thead>
<tr>
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<tr>
<td>( x = r \cos \theta )</td>
<td>( r = \rho \sin \phi )</td>
<td>( x = \rho \sin \phi \cos \theta )</td>
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<td>( y = r \sin \theta )</td>
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\[ dV = dx \, dy \, dz = dz \, r \, d\theta \, dr = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

Substitutions in multiple integrals
\[ \int \int_R f(x, y) \, dx \, dy = \int \int_G f(x(u, v), y(u, v)) \left| J(u, v) \right| \, du \, dv \quad \text{where} \quad J(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \quad \left| \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v} \right| \]

Mass, moments, and center of mass
Mass \( M = \int \int_R \delta \, dA \)

Moments \( M_x = \int \int_R y \, \delta \, dA \) \quad \( M_y = \int \int_R x \, \delta \, dA \) \quad Center of mass \( \bar{x} = M_y / M \) \quad \( \bar{y} = M_x / M \)

Green's Theorem in the \( x-y \) plane (The curve \( C \) is traversed counterclockwise, and \( \mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} \).)

Circulation \[ \oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \int \int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_C (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA = \int \int_C (Q_x - P_y) \, dA \]

Outward Flux \[ \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int \int_R \nabla \cdot \mathbf{F} \, dA = \int \int_R (P_x + Q_y) \, dA \]

Surface area of level surface \( g(x, y, z) = c \)
\[ SA = \int \int_S \delta \, ds = \int \int_R \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} \, dA \]

Stokes' Thm: \[ \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, ds \quad \text{Divergence Thm:} \quad \int \int_S \mathbf{F} \cdot \mathbf{n} \, ds = \int \int_C \nabla \cdot \mathbf{F} \, dV \]

Fond memories
\[ \sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2} \]