\[ f(x, y) = 10 + \frac{y^2}{2} + x^2(1-y) \]

(a) \( f_x = 2x(1-y) \) \quad \( f_y = y - x^2 \)

So critical points are located at the solution to:

1. \( 2x(1-y) = 0 \) \quad AND \quad 2. \( y - x^2 = 0 \)

\[ \downarrow \]

\[ x = 0 \]

or

\[ y = 1 \]

Case 1: \( x = 0 \)

Plug \( x = 0 \) into equation 2:

\[ y - 0 = 0 \Rightarrow y = 0 \]

\( \Rightarrow \) CP at \( (0,0) \)

Case 2: \( y = 1 \)

Plug \( y = 1 \) into equation 2:

\[ 1 - x^2 = 0 \Rightarrow x = \pm 1 \Rightarrow \text{CPs at } (1,1) \text{ and } (-1,1) \]

OVER
<table>
<thead>
<tr>
<th>point</th>
<th>function value</th>
<th>$f_{xx}$</th>
<th>$f_{yy}$</th>
<th>$f_{xy}$</th>
<th>$D$</th>
<th>classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>10</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>local minimum</td>
</tr>
<tr>
<td>(1,1)</td>
<td>$10 + \frac{1}{2}$</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>-4</td>
<td>saddle point</td>
</tr>
<tr>
<td>(-1,1)</td>
<td>$10 + \frac{1}{2}$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>-4</td>
<td>saddle point</td>
</tr>
</tbody>
</table>

$$f_{xx} = 2(1-y)$$

$$f_{yy} = 1$$

$$D = f_{xx}f_{yy} - f_{xy}^2$$

$$f_{xy} = -2x$$

(b) $f(0,0) = 10$

$f(-1,1) = 10 + \frac{1}{2}$

$f(1,1) = 10 + \frac{1}{2}$

(c) $(0,0)$ is the only stable equilibrium, if the marble came to rest at a saddle point, any small bump/movement of rate would set the marble in motion.
\[ V(r, h) = \pi r^2 h \]

- we measure \( r \) and \( h \) with relative errors \( \Delta r / r \) and \( \Delta h / h \) respectively, resulting in a relative error, \( \Delta V / V \), in the volume we calculate based on these measurements.
- want \( \Delta V / V < 0.01 \)
- \( \frac{\Delta h}{h} = 2 \frac{\Delta r}{r} \) 
  "relative error in height is twice the relative error in radius"

\[ \Delta V \approx dV = \frac{\partial V}{\partial r} \Delta r + \frac{\partial V}{\partial h} \Delta h \]

\[ = 2\pi rh \Delta r + \pi r^2 \Delta h \]

\[ \Rightarrow \frac{\Delta V}{V} = \frac{2\pi rh \Delta r}{\pi r^2 h} + \frac{\pi r^2 \Delta h}{\pi r^2 h} \]

\[ = 2 \frac{\Delta r}{r} + \frac{\Delta h}{h} \]

\[ = \frac{\Delta h}{h} + \frac{\Delta h}{h} \]

\[ = 2 \frac{\Delta h}{h} \]

\[ \Rightarrow \frac{\Delta h}{h} \text{ can be no bigger than } 0.005 \]

\[ \frac{\Delta r}{r} \text{ can be no bigger than } 0.0025 \]

(And \( \frac{\Delta r}{r} = \frac{1}{2} \frac{\Delta h}{h} \) exactly, otherwise the above work isn't applicable.)
#3

\( \mathbf{R}(t) \): Bonnie's path

\( T(x,y) \): temperature distribution

\( t^* \): a particular time

\[ \mathbf{R}(t^*) = \langle 1, 3 \rangle \]

\[ \nabla T(t^*) = \langle 2, 1 \rangle \]

\[ \nabla T \big|_{t^*} = \langle 2, 5 \rangle \]

\[ \frac{d}{dt} \mathbf{R}(t^*) = \langle 3, 2 \rangle \]

This means "evaluate at \( t = t^* \)"

(a) \[
\frac{dT}{dt} \bigg|_{t^*} = \left( \frac{dT}{dx} \frac{dx}{dt} + \frac{dT}{dy} \frac{dy}{dt} \right) \bigg|_{t^*} = \langle T_x, T_y \rangle \cdot \langle \dot{x}/\dot{t}, \dot{y}/\dot{t} \rangle \bigg|_{t^*}
\]

\[ = \nabla T \cdot \nabla T(t^*) = \langle 2, 5 \rangle \cdot \langle 2, 1 \rangle = 4 + 5 = 9 \]

(b) \[
\frac{dT}{ds} \bigg|_{t^*} = \left( \nabla T \cdot \hat{\mathbf{u}} \right) \bigg|_{t^*} = \left( \nabla T \cdot \frac{\nabla T}{|\nabla T|} \right) \bigg|_{t^*} = \left( \frac{dT}{ds} \frac{1}{|\nabla T|} \right) \bigg|_{t^*} = \frac{9}{\sqrt{5}}
\]

Bonnie's flying in the direction \( \nabla T/|\nabla T| = \hat{t} \)

(c) \( \Delta t = 0.1 \)

\[ \Delta T \approx \frac{dT}{dt} \Delta t = 9(0.1) = 9 \]

(d) dir. of greatest mc. = parallel to \( \nabla T = \langle 2, 5 \rangle \)

In (b), we showed that \( \frac{dT}{ds} = |\nabla T| \frac{dt}{ds} \). In our new direction, we have that \( \frac{dT}{ds} = \nabla T \cdot \frac{\nabla T}{|\nabla T|} = |\nabla T|, \) so
\[ \frac{dT}{dt} = 101 11 \]

At \( t = t^* \), this gives us \( \frac{dT}{dt} = \sqrt{14+25} = \sqrt{39} \), so finally if \( \Delta t = 0.1 \), then

\[ \Delta T \approx \frac{dT}{dt} \Delta t = \sqrt{39} \]
Objective: minimize the total anger level, which is given by

\[ A(\alpha, \beta, \gamma) = \frac{1}{2} \left[ (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 + \alpha^2 + \beta^2 + \gamma^2 \right] \]

Constraint: the angles \( \alpha, \beta, \gamma \) must add up to \( 2\pi \) because they make a whole circle:

\[ g(\alpha, \beta, \gamma) = \alpha + \beta + \gamma = 2\pi \]

**doing this in degrees gives the same angles, but A turns out to be \( \frac{3}{2}(120^2) \) which should be \( \frac{3}{2}(360^2) \) radians**

\[ \nabla A = \lambda \nabla g \]

\[ \Rightarrow \langle \alpha - \beta - (\beta - \alpha) + \alpha, -(\alpha - \beta) + \beta - \gamma + \beta, -(\beta - \gamma) + \gamma - \alpha + \gamma \rangle \]

\[ = \lambda \langle 1, 1, 1 \rangle \]

1. \( 3\alpha - \beta - \gamma = \lambda \)
2. \( -\alpha + 3\beta - \gamma = \lambda \)
3. \( -\alpha - \beta + 3\gamma = \lambda \)

Plugging \( \alpha = \beta \) and \( \alpha = \gamma \) into the constraint:

\[ 3\alpha = 2\pi \Rightarrow \alpha = \beta = \gamma = \frac{2\pi}{3} \] with \( A = \frac{3}{2} \left( \frac{2\pi}{3} \right)^2 = \frac{2\pi^2}{3} \)

We know this is a minimum, not a maximum, because if we plug \( \alpha = \beta = \gamma \) into \( A \), we get \( A(\alpha, \alpha, \alpha) = \frac{3}{2} \alpha^2 \) which is concave up at every angle \( \alpha \).
\[ f(x, y) = \frac{y^3}{3} + \frac{x^3}{3} (1 - y) \]

\[ f_x = x^2 (1 - y) \]

\[ f_y = y^2 - \frac{x^3}{3} \]

\[ f_{xx} = 2x (1 - y) \]

\[ f_{yy} = 2y \]

\[ f_{xy} = -x^2 \]

(a) Linearization about (1, 1)

\[ f(x, y) \approx f(1, 1) + f_x(1, 1)(x-1) + f_y(1, 1)(y-1) \]

\[ \Rightarrow f(x, y) \approx \frac{1}{3} + \phi(x-1) + \frac{2}{3}(y-1) = \frac{2}{3}y - \frac{1}{3} \]

(b) \[ f(0.9, 1.1) \approx \frac{1}{3} + \frac{2}{3}(0.1-1) \]

\[ \Rightarrow f(0.9, 1.1) \approx \frac{1}{3} + \frac{2}{3}(.1) \]

(c) \[ |x-1| \leq .2 \]

\[ |y-1| \leq .2 \]

\[ |E(x, y)| \leq \frac{1}{2} M (|x-x_0| + |y-y_0|)^2 \]

where \[ M = \max_R \left\{ |f_{xx}|, |f_{yy}|, |f_{xy}| \right\} \]

\[ |f_{xx}| = |2x (1 - y)| = 2|x| |1 - y| \]

\[ \Rightarrow |f_{xx}| \leq 2(1.2)(1.1) \text{ in } R \]

\[ |f_{yy}| = |2y| = 1.2(.4) \]

\[ \Rightarrow |f_{yy}| \leq 2(1.2) \text{ in } R \]

\[ |f_{xy}| = x^2 \]

\[ \Rightarrow |f_{xy}| \leq 1.2^2 \text{ in } R \]
Since \( -4(1.2) < 1.2(1.2) < 2(1.2) \)
\[
\frac{\max |f_{xx}|}{\max |f_{yy}|} \leq \max |f_{xy}|
\]
we pick \( M = 2(1.2) = 2.4 \)

\[
|E(x,y)| \leq \frac{1}{2} (2.4) (|x-x_0| + |y-y_0|)^2
\]
\[
\leq .2 \; \text{in} \; R \quad \leq .2 \; \text{in} \; R
\]

\[
|E(x,y)| \leq \frac{1}{2} (2.4) (-4)^2 = \frac{1}{2} (2.4) .16 = 2.4 (.08) = 1.2 (.16)
\]