INSTRUCTIONS: Electronic devices are not permitted during the exam. Write your name, your instructor's name, and your recitation number on the front of your bluebook. Start each problem on a new right-hand page. Justify your work clearly and box your final answer. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (25 points) Consider the two points \( P_1(1,0,2) \) and \( P_2(-1,2,0) \) and the plane \( M \) described by the standard equation \( 2x - 4y + z = 10 \).

(a) Determine the value of \( c \) such that the point \( P_3(1/2,1/4,c) \) lies in the plane \( M \).

(b) Find the standard equation for a new plane \( N \) containing the points \( P_1 \) and \( P_2 \), that is perpendicular to the plane \( M \).

(c) Find the parameterization for the line orthogonal to \( N \), that lies in the plane \( M \), and also passes through \( P_3 \).

2. (25 points) Starting from time \( t = 0 \), an object's position in space is given by the vector

\[
\mathbf{r}(t) = \sin \left( t^2 \right) \mathbf{i} + \cos \left( t^2 \right) \mathbf{j} + t^2 \mathbf{k}.
\]

(a) Calculate the object's speed, \( |\mathbf{v}| \), and its unit tangent vector \( \mathbf{T} \), for any time \( t \geq 0 \). (Tip: be sure to check your work here because the rest of the problems depends on these results!)

(b) Calculate the arc length, \( s(t) \), for any time \( t \geq 0 \).

(c) Calculate the unit vector \( \mathbf{N} \) for any time \( t \geq 0 \).

(d) Calculate the unit vector \( \mathbf{B} \) for any time \( t \geq 0 \).

(e) What is the curvature of the particle's path for any time \( t \geq 0 \)?

3. (25 points) Determine functions \( x(t) \), \( y(t) \), and \( z(t) \), such that the parameterization of the path \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \) satisfies the following descriptions:

(a) The line through the points \( (1,2,3) \) and \( (3,2,1) \).

(b) The curve \( x - y^2 \) in the \( z = 2 \) plane.

(c) The curve \( y^2 + z^2/4 = 1 \) in the \( x = 2 \) plane.

(d) The intersection of the surfaces \( z = 2 - x^2 - y^2 \) and \( z = y^2 - x^2 \), for \( y > 0 \).

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4. (25 points) For each of the surfaces described below (a through d), list the possible equations (1 through 8) that would fit the description. Note that there might be more than one equation that fits the description. Also, there might not be any equations that fit the description, in which case you should state NONE.

(a) Paraboloid parallel to the y-axis.
(b) A set of cones parallel to the x-axis.
(c) Hyperboloid of two sheets parallel to the z-axis.
(d) Hyperboloid of one sheet parallel to the y-axis.

(1) \( \frac{x^2}{4} + \frac{z^2}{9} + 9 = \frac{y}{9} \)
(2) \( \frac{y^2}{16} + \frac{z^2}{4} = \frac{x^2}{9} \)
(3) \( \frac{x^2}{4} + \frac{z^2}{16} = \frac{y}{9} \)
(4) \( \frac{y^2}{16} + \frac{z^2}{4} = \frac{x^2}{25} - 1 \)
(5) \( \frac{x^2}{16} - \frac{y^2}{9} + \frac{z^2}{4} = 1 \)
(6) \( \frac{y^2}{x} = \frac{z^2}{4} \)
(7) \( \frac{x^2}{16} + \frac{y^2}{9} = \frac{z^2}{4} - 1 \)
(8) \( \frac{x^2}{4} = \frac{y^2}{5} + \frac{z^2}{9} \)

Projections, and distances from a point to a line, and a point to a plane

\[
\text{proj}_A B = \left( \frac{A \cdot B}{A \cdot A} \right) A \quad d = \frac{|\vec{P_0} \times \vec{v}|}{|\vec{v}|} \quad d = \left| \frac{|\vec{P_0} \cdot \vec{n}|}{|\vec{n}|} \right|
\]

Arc length, Frenet formulas, and tangential and normal acceleration components

\[
ds = |\vec{v}| \, dt \quad \vec{T} = \frac{d\vec{r}}{ds} = \frac{\vec{v}}{|\vec{v}|} \quad \vec{N} = \frac{d\vec{T}/ds}{|d\vec{T}/ds|} = \frac{d\vec{T}/dt}{|d\vec{T}/ds|} \quad \vec{B} = \vec{T} \times \vec{N} \]
\[
\frac{d\vec{T}}{ds} = \kappa \vec{N} \quad \frac{d\vec{B}}{ds} = -\tau \vec{N} \quad \kappa = \frac{|d\vec{T}|}{ds} = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{|f''(x)|}{\sqrt{1 + (f'(x))^2}}^{3/2} = \frac{|\dot{x} \ddot{y} - \ddot{x} \dot{y}|}{(\dot{x}^2 + \ddot{y})^{3/2}} \quad \tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}
\]
\[
\vec{a} = a_N \vec{N} + a_T \vec{T} \quad \text{where} \quad a_T = \frac{d|\vec{v}|}{dt} \quad \text{and} \quad a_N = \kappa |\vec{v}|^2 = \sqrt{|\vec{a}|^2 - a_T^2}
\]
#1

$P_1 (1,0,2) \quad P_2 (-1,2,0)$

$M: 2x - 4y + z = 10$

(a) $P_3 (\frac{1}{2}, \frac{1}{4}, c)$

For this point to lie on the plane $M$, it must satisfy the equation for $M$:

$$\frac{2\left(\frac{1}{2}\right) - 4\left(\frac{1}{4}\right) + c}{1} = 10 \Rightarrow c = 10$$

(b) Since $N \perp M$, the normal vector to $M$, $
\vec{n} = \langle 2, -4, 1 \rangle$ (from the eq. for $M$)

is parallel to $N$. Since $P_1$ and $P_2$ lie in $N$ (as well as $M$), the vector $\vec{P_1P_2}$ is also parallel to $N$. This means that $\vec{P_1P_2} \times \vec{n}$ is normal to the plane $N$. So...

$$\vec{P_1P_2} = -2\hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{P_1P_2} \times \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & -2 \\ 2 & -4 & 1 \end{vmatrix} = \langle 2 - 8, 2 - 4, 8 - 4 \rangle = \langle -6, -2, 4 \rangle$$

$$= 2 \langle -3, -1, 2 \rangle$$
$-3(x-1) - y + 2(z-2) = 0$

(C) \parallel \overrightarrow{P_2P_3} \times \overrightarrow{n}$ (i.e. \langle -3, -1, 2 \rangle)

So the line is parallel to \( \overrightarrow{v} = \langle -3, -1, 2 \rangle \), and passes through \( P_3 (\frac{1}{2}, 4, 10) \), giving a parameterization of

\[
\overrightarrow{r}(t) = \langle \frac{1}{2}, 4, 10 \rangle + t\langle -3, -1, 2 \rangle = \langle \frac{1}{2} - 3t, 4 - t, 10 + 2t \rangle
\]

\[\boxed{\# 2}\]

\( \overrightarrow{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle \) \( t > 0 \)

(a) \( \overrightarrow{v}(t) = \langle 2t \cos(t^2), -2t \sin(t^2), 2t \rangle \)

\[
|\overrightarrow{v}(t)| = \sqrt{4t^2 \cos^2(t^2) + 4t^2 \sin^2(t^2) + 4t^2} = \sqrt{4t^2(\sin^2(t^2) + \cos^2(t^2)) + 4t^2} = \sqrt{4t^2 + 4t^2} = 2|t|\sqrt{2} = |\overrightarrow{v}(t)| \quad \text{(don't need absolute value since } t > 0)\]

\[
\hat{t} = \frac{1}{2|t|} \langle 2t \cos(t^2), -2t \sin(t^2), 2t \rangle
\]

but since \( t > 0 \), \( |t| = t \), so

\[
\hat{t} = \frac{1}{2t} \langle \cos(t^2), -\sin(t^2), 1 \rangle \quad \text{for } t > 0.
\]
(b) \( S(t) = \int_0^t |\dot{\mathbf{r}}(\tau)| \, d\tau = \int_0^t 2\sqrt{2} \tau \, d\tau = \sqrt{2} \tau^2 \bigg|_0^t = \sqrt{2} t^2 \)

\[ \Rightarrow S(t) = \sqrt{2} t^2 \quad t > 0 \]

(c) \[ \hat{N}(t) = \frac{\hat{T}(t)}{|\frac{dT}{dt}|} \]

\[ \frac{dT}{dt} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2t \sin(t^2) \\ -2t \cos(t^2) \\ 0 \end{pmatrix} \]

\[ |\frac{dT}{dt}| = \frac{1}{\sqrt{2}} \sqrt{4t^2 \sin^2 t^2 + 4t^2 \cos^2 t^2} = \frac{1}{\sqrt{2}} 2|t| = \frac{2t}{\sqrt{2}} \quad \text{for} \ t > 0 \]

\[ \Rightarrow \hat{N} = \begin{pmatrix} -\sin(t^2) \\ -\cos(t^2) \\ 0 \end{pmatrix} \quad t > 0 \]

(d) \[ \hat{B} = \hat{T} \times \hat{N} \]

\[ = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{2}} \cos(t^2) & -\frac{1}{\sqrt{2}} \sin(t^2) & \frac{1}{\sqrt{2}} \\ -\sin(t^2) & -\cos(t^2) & 0 \end{vmatrix} \quad t > 0 \]

\[ = \begin{pmatrix} \frac{1}{\sqrt{2}} \cos(t^2) \\ -\frac{1}{\sqrt{2}} \sin(t^2) \\ -\frac{1}{2} \cos^2(t^2) - \frac{1}{2} \sin^2(t^2) \end{pmatrix} \]

\[ \hat{B}(t) = \begin{pmatrix} \frac{1}{\sqrt{2}} \cos(t^2) \\ -\frac{1}{\sqrt{2}} \sin(t^2) \\ -\frac{1}{2} \end{pmatrix} \quad t > 0 \]

(e) \[ k = \left| \frac{dT}{ds} \right| = \left| \frac{dT}{dt} \frac{dt}{ds} \right| = \left| \frac{dT}{dt} \right| \left| \frac{1}{N} \right| = \frac{2t}{\sqrt{2}} \frac{1}{2t} = \frac{1}{2} \]
(a) \( P(1,2,3) \quad Q(3,2,1) \)

\[ \vec{v} = \vec{PQ} = \langle 2, 0, -2 \rangle \]

\[ \Rightarrow \vec{r}(t) = \langle 1, 2, 3 \rangle + t \langle 2, 0, -2 \rangle = \langle 1 + 2t, 2, 3 - 2t \rangle \]

OR

\[
\begin{align*}
\text{x}(t) &= 1 + 2t \\
\text{y}(t) &= 2 \\
\text{z}(t) &= 3 - 2t
\end{align*}
\]

(b) \( x = y^2 \quad z = 2 \)

We can choose \( y = t \), then we must have \( x = t^2 \) for \( x = y^2 \) to be satisfied. \( z \) is simply fixed at 2.

\[ \Rightarrow \vec{r}(t) = \langle t^2, t, 2 \rangle \]

OR

\[
\begin{align*}
\text{x}(t) &= t^2 \\
\text{y}(t) &= t \\
\text{z}(t) &= 2
\end{align*}
\]
(c) \( y^2 + z^2 / 4 = 1 \) \( x = 2 \)

This is an ellipse in \( y \) and \( z \):

\[
\begin{align*}
y(t) &= \cos(t) \\
z(t) &= 2 \sin(t) \\
x(t) &= 2
\end{align*}
\]

(d) \( z = 2 - x^2 - y^2 \) \( z = y^2 - x^2 \) \( y > 0 \)

(1) eliminate a variable:

\[
2 - x^2 - y^2 = z = y^2 - x^2
\]

\[\Rightarrow 2 - x^2 - y^2 = y^2 - x^2 \Rightarrow z = 2y^2 \Rightarrow y = 1 \quad (y > 0)\]

\[\Rightarrow \text{the shadow in the xy plane is the line } y = 1\]

(2) plug \( y = 1 \) into either equation:

\[z = 1 - x^2\]

\[\Rightarrow \text{choose } x = t, \text{ then we must have } z = 1 - t^2\]

\[
\begin{align*}
x(t) &= t \\
y(t) &= 1 \\
z(t) &= 1 - t^2
\end{align*}
\]
(1) \(-\frac{x^2}{4} + \frac{y}{q} - \frac{z^2}{q} = 9\)

\[ x = 0 \quad \frac{y}{q} \quad \frac{z^2}{q} = 9 \]

\[ y = 81 + 2^2 \quad \text{a parabola opening left} \]

\[ -\frac{x^2}{4} - \frac{z^2}{q} = 9 \quad \Rightarrow \text{no solution, doesn't cross this plane} \]

\[ y = 90 \]

\[ 1 = \frac{x^2}{4} + \frac{y^2}{q} \]

\[ z = 0 \quad \text{a parabola opening up} \]

\[ y = 81 + \frac{9}{4}x^2 \]

paraboloid \( \parallel \) to y-axis (a)
(2) \[-\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0\]

\[x = 0\]
\[\frac{y^2}{b} + \frac{z^2}{c} = 0\]
\[\frac{x^2}{a} = \frac{z^2}{c}\]
\[\frac{x^2}{a} = \frac{y^2}{b}\]

\[x = 3\]
\[\frac{y^2}{b} + \frac{z^2}{c} = 1\]

set of cones \(\parallel\) to \(x\)-axis (6)

(3) just like (1), but not shifted away from the origin

\(\Rightarrow\) paraboloid \(\parallel\) to \(y\)-axis (a)

(4) \[\frac{x^2}{25} - \frac{y^2}{4} - \frac{z^2}{100} = 1\]

\[x = 0\]
\[-\frac{y^2}{4} - \frac{z^2}{100} = 1\]

\[x = 0\]
\[\frac{x^2}{25} - \frac{z^2}{100} = 1\]

\[\frac{y^2}{4} = 1\]
\[\frac{x^2}{25} - \frac{y^2}{4} = 1\]
hyperboloid of 2 sheets (c) parallel to x-axis

\[ \frac{x^2}{16} - \frac{y^2}{9} + \frac{z^2}{4} = 1 \]

\[ \frac{x^2}{16} + \frac{z^2}{4} = 1 \]

\[ \frac{x^2}{16} - \frac{y^2}{9} = 1 \]

hyperboloid of 1 sheet parallel to y-axis (d)
(6) \[ \frac{x}{a} + \frac{y^2}{2a} + \frac{z^2}{4} = 0 \]

\[ \frac{x}{a} = \frac{y^2}{2a} = \frac{z^2}{4} \]

\[ \frac{x}{a} = \frac{y^2}{2a} \]

\[ \frac{y}{x} = \frac{z^2}{a} \]

\[ \frac{z}{x} = \frac{y^2}{a} \]

\[ \frac{x}{a} = \frac{y^2}{2a} \]

**hyperbolic paraboloid**

(7) like (6), but \( x \leftrightarrow z \) \( \Rightarrow \) hyperboloid of 2 sheets parallel to \( z \)-axis

(8) \[ \frac{x}{a} + \frac{y^2}{a} - \frac{z^2}{4} = 0 \]

\[ \frac{x}{a} = \frac{y^2}{a} = \frac{z^2}{4} \]

\[ \frac{x}{a} = \frac{y^2}{a} \]

\[ \frac{y}{x} = \frac{z^2}{a} \]

\[ \frac{z}{x} = \frac{-y^2}{a} \]

**hyperbolic paraboloid**
(a) 1, 3
(b) 2
(c) 4
(d) 5

NONE 6, 7, 8