1. (30 pts) Determine whether the following series are conditionally convergent, absolutely convergent, or divergent. You may NOT use the Ratio or Root Test for this problem. State any tests you use.

(a) \[ \sum_{n=3}^{\infty} \frac{\sqrt{n^2 + 1}}{n^5 - n} \]

Solution:

(a) Looks like the series with \( a_n = \frac{\sqrt{n^2}}{n^5} = \frac{1}{n^{\frac{9}{2}}} \), which is a convergent p-series. But our series is greater than a convergent series, so the Direct Comparison Test does not work. So we turn to the Limit Comparison Test.

First, note that our series terms and those of \( \sum \frac{1}{n^3} \) are always positive, so the LCT applies. Now then:

\[
\lim_{n \to \infty} \frac{\frac{1}{n^\frac{9}{2}}}{\frac{1}{n^3}} = \lim_{n \to \infty} \frac{n^\frac{9}{2}}{n^3} = \lim_{n \to \infty} \frac{1}{n^{\frac{3}{2}}} = 0
\]

So the series is convergent. Because the terms are all positive, \( \sum |a_n| = \sum a_n \), and regular convergence is the same as absolute convergence. Therefore, \textbf{ABSOLUTELY CONVERGENT by LIMIT COMPARISON TEST}.

(b) This is a geometric series in disguise:

\[ \sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}} = \sum_{n=1}^{\infty} \frac{(-3)^2(-3)^{n-1}}{4^{n-1}} = \sum_{n=1}^{\infty} 9 \left(\frac{-3}{4}\right)^{n-1}, \]

which is a geo. series with \( a = 9 \) and \( r = -\frac{3}{4} \). \( |r| = \frac{3}{4} < 1 \), so the series converges.

To check absolute convergence, note that \( \sum |a_n| \) is just a geo. series with \( r = \frac{3}{4} < 1 \), which is convergent. Therefore, \textbf{ABSOLUTELY CONVERGENT by GEO SERIES}.

(c) We will begin by testing for absolute convergence, using \( \sum_{n=2}^{\infty} \frac{|(-1)^n|}{2n \ln(n)} \). We will use the integral test and compare with \( \int_{2}^{\infty} \frac{1}{2x \ln x} \, dx = \lim_{t \to \infty} \int_{2}^{t} \frac{1}{2x \ln x} \, dx \). First, note that \( f(x) = \frac{1}{2x \ln x} \) is continuous (for \( x > 0 \)), positive (for \( x > 1 \)) and decreasing (\( b_{n+1} = \frac{1}{2(n+1) \ln(n+1)} < \frac{1}{2n \ln n} = b_n \), so the Integral Test applies. Using u-substitution, with \( u = \ln(x) \) and \( du = \frac{1}{x} \, dx \), we get

\[
\lim_{t \to \infty} \int_{2}^{t} \frac{1}{2u} \, du = \lim_{t \to \infty} \frac{1}{2} \ln(u) \bigg|_{2}^{t} = \lim_{t \to \infty} \frac{1}{2} (\ln(t) - \ln(2)).
\]

This integral diverges! So we know that the series \( \sum_{n=2}^{\infty} \frac{|(-1)^n|}{2n \ln(n)} \) is \textbf{NOT} absolutely convergent.

Now we test conditional convergence using the AST. The series is alternating, \( \lim_{n \to \infty} \frac{1}{2n \ln(n)} = 0 \) and \( \frac{1}{2(n+1) \ln(n+1)} \leq \frac{1}{2n \ln(n)} \) for all \( n > 2 \). So we can apply the AST and the series is \textbf{CONDITIONALLY CONVERGENT BY AST and INTEGRAL TEST}. 
2. (10 pts) Suppose the $n^{th}$ partial sum of a series $\sum_{n=1}^{\infty} a_n$ is given by $s_n = \frac{1}{n+1} + \frac{1}{n}$.

(a) Determine an expression for $a_n$.

(b) Does the series converge? If so, to what?

Solution:

(a) $a_n = s_n - s_{n-1} = \left( \frac{1}{n+1} + \frac{1}{n} \right) - \left( \frac{1}{n} + \frac{1}{n-1} \right) = \frac{1}{n+1} - \frac{1}{n-1}$

(b) Series convergence is defined by $\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{1}{n+1} + \frac{1}{n} = 0$, therefore the series **CONVERGES** to 0.

3. (a) (15 pts) Consider the function $f(x) = \sum_{n=3}^{\infty} \frac{4^n x^{2n}}{n-1}$. What is the interval of convergence of $f(x)$?

(b) (5 pts) Using power series, one can find $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, a series which converges for all $x$.

Use the first 2 terms of this series representation for $\cos x$ to come up with an approximation for $\sqrt{2}$.

(Hint: You may find it helpful to know that $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$.)

(c) (8 pts) Using the Alternating Series Error Estimation Theorem, give a reasonable bound for the error in our approximation in part (b) above. Be sure to justify using this theorem!

Solution:

(a) Ratio test:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{\frac{4^{n+1} x^{2(n+1)}}{(n+1)-1}}{\frac{4^n x^{2n}}{n-1}} = \lim_{n \to \infty} \frac{4^{n+1} x^{2n+2}}{n-1} \times \frac{n-1}{4^n x^{2n}} = \lim_{n \to \infty} 4 \left( 1 - \frac{1}{n} \right) x^2 \]

\[= 4x^2 < 1 \text{ for convergence } \rightarrow -\frac{1}{2} < x < \frac{1}{2} \]

Now check endpoints:

$x = -\frac{1}{2}$: Have

\[\sum_{n=3}^{\infty} \frac{4^n (-\frac{1}{2})^{2n}}{n-1} = \sum_{n=3}^{\infty} \frac{4^n (-\frac{1}{2})^2}{n-1} = \sum_{n=3}^{\infty} \frac{4^n (\frac{1}{2})}{n-1} = \sum_{n=3}^{\infty} \frac{1}{n-1} \]

Now $0 \leq \frac{1}{n-1}$, therefore the $x = -\frac{1}{2}$ endpoint is **divergent** by Direct Comparison Test.

When $x = \frac{1}{2}$, the exact same work applies because of the even powers of $x$. Therefore, the $x = \frac{1}{2}$ endpoint is also **divergent** by Direct Comparison Test.

The interval of convergence then is $\left[ -\frac{1}{2} < x < \frac{1}{2} \right]$.

(b) The first 2 terms of the series gives the approximation $\cos x \approx 1 - \frac{x^2}{2!} = 1 - \frac{x^2}{2}$

Thus, $\cos \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \approx 1 - \frac{(\frac{\pi}{4})^2}{2}

\rightarrow \sqrt{2} \approx 2 - \frac{\pi^2}{16}$

(c) To use the Alternating Series Error Estimation Theorem, we need to know that for our alternating series, $\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{4})^{2n}}{(2n)!}$, that $b_n$ is (1) decreasing and (2) has limit 0. So with $b_n = \frac{(\frac{\pi}{4})^{2n}}{(2n)!}$, we see
that
\[ b_{n+1} = \frac{\left(\frac{\pi}{4}\right)^{2(n+1)}}{(2(n+1))!} = \frac{\left(\frac{\pi}{4}\right)^2}{(2n+1)!} b_n < b_n \] so it is decreasing

(2) \[ \lim_{n \to \infty} \frac{\left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = 0 \]

so the Alternating Series Error Estimation Theorem applies. Therefore the error is less than the first unused term, which is \( \left(\frac{\pi}{4}\right)^4 \frac{4!}{24} \). Note, however, that we multiplied by 2 in order to obtain our approximation in (b). So another acceptable answer is \( \left(\frac{\pi}{2}\right)^4 12 \)

4. (a) (10 pts) Find the power series representation of \( f(x) = \frac{x}{9 + x^2} \). Do not find the interval of convergence.

(b) (10 pts) Find the closed form function represented by the series \( \sum_{n=1}^{\infty} n(5x)^{n-1} \) for \( |x| < \frac{1}{5} \).

Solution: (a) We want this to match the form of the power series sum, \( \frac{1}{1-x} \). To do this, we manipulate \( f(x) \).

\[
 f(x) = \frac{x}{9 + x^2} = \frac{x}{9} \frac{1}{1 + \left(-\frac{x}{3}\right)^2} = \frac{x}{9} \frac{1}{1 - \left(-\frac{x}{3}\right)^2}
\]

We know that \( \frac{1}{1 - \left(-\frac{x}{3}\right)^2} = \sum_{n=0}^{\infty} \left[ \frac{x}{3} \right]^n \).

So
\[
 \left(\frac{x}{9}\right) \frac{1}{1 - \left(-\frac{x}{3}\right)^2} = \frac{x}{9} \sum_{n=0}^{\infty} \left[ \frac{x}{3} \right]^n.
\]

\[
 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{9n+1}
\]

(b) This is the derivative of the power series \( \frac{1}{5} \sum_{n=0}^{\infty} (5x)^n \). We know this series represents the function \( \frac{1}{1 - 5x} \). So we take the derivative of that function, \( \left(\frac{1}{5}\right) \frac{d}{dx} \left(\frac{1}{1 - 5x}\right) \).

\[
 \left(\frac{1}{5}\right) \frac{1}{(1 - 5x)^2}
\]

5. (12 pts) Short answer

(a) Consider the power series \( \sum_{n=0}^{\infty} c_n (x - 2)^n \). If we know that \( \sum_{n=0}^{\infty} c_n 5^n \) converges, what can we say about the convergence of the following? Justify your answers in 1-2 sentences.

(i) \( \sum_{n=0}^{\infty} c_n (-2)^n \)  
(ii) \( \sum_{n=0}^{\infty} c_n (7)^n \)

(b) Given the sequence of partial sums, \( \{s_n\}_{n=1}^{\infty} \), of the series \( \sum_{n=1}^{\infty} a_n \), suppose \( \lim_{n \to \infty} s_n = 2 \).

(i) What can be said about the convergence/divergence of the series \( \sum_{n=1}^{\infty} a_n \)?

(ii) What can be said about the convergence/divergence of sequence \( \{a_n\}_{n=1}^{\infty} \)?
Solution:
(a) Given that the radius of convergence is at least 5, we know that for \( x \) values satisfying \( |x - 2| < 5 \) the series must converge. Solving this for the interval, we find that \(-3 < x < 7\). We do not know whether we have convergence on the endpoints.

(i) \( \sum_{n=0}^{\infty} c_n (-2)^n \) converges because \( x - 2 = -2 \to x = 0 \) and \(-3 < 0\)

(ii) Whether \( \sum_{n=0}^{\infty} c_n (7)^n \) converges or diverges is unknown because \( x - 2 = 7 \to x = 9 \) is outside the known radius of convergence, but could be within the actual radius; we would need more information to know.

(b)

(i) By definition of series convergence, you know that \( \sum a_n = \lim s_n = 2 \). So the series converges to 2

(ii) From the Divergence Test (or the theorem we got it from), if a series converges, then the sequence must have \( \lim a_n = 0 \). So the sequence converges to 0