1. (a) (6 pts) Find the Maclaurin series for \( f(x) = \frac{\sin(x^2)}{x} \). What is the radius of convergence?
(b) (4 pts) What is the 5\(^{th}\)-order Taylor polynomial for \( f(x) \) centered at \( x = 0 \)?
(c) (6 pts) Use the 3\(^{rd}\)-order Taylor polynomial for \( e^x \) centered at \( x = 0 \) to find an approximation to \( e \).
(d) (8 pts) Find upper bounds on the errors in the approximations in (b) and (c), if \( 0 \leq x \leq 2 \). (You do not need to justify using these theorems.)

Solution:

(a) Use the given series \( \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \), \( R = \infty \) to find
\[
\frac{1}{x} \sin(x^2) = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n+1)!}
\]
with \( R = \infty \) since we had \( |x^2| < \infty \).

(b) Just write out terms from the above Maclaurin series until you hit \( x^5 \):
\[
T_5 = 1 - \frac{x^5}{3!} = 1 - \frac{x^5}{6}
\]

(c) Given \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \)
\[
\rightarrow e = e^1 \approx 1 + (1) + \frac{(1^2)}{2} + \frac{(1^3)}{6} = 2 + \frac{2}{3} = \frac{8}{3}
\]

(d) For (b), use Alternating Series Error: \( |\text{error}| \leq \text{first unused term} = \frac{x^9}{9!} \leq \frac{(2)^9}{120} = \frac{64}{15} \)
For (c), use Taylor’s Formula: \( |\text{error}| \leq \left| \frac{f^{(n+1)}(z)}{(n+1)!} (x - 0)^{n+1} \right| = \left| \frac{e^z x^4}{4!} \right| \leq \frac{e^2 (2)^4}{24} = \frac{2e^2}{3} \)

2. Consider the integral \( \int_{0}^{4} e^{-x^2} \, dx \).
(a) (4 pts) Find an approximation to this integral, \( M_4 \), using the Midpoint Rule and \( n = 4 \) subintervals. Do not simplify your answer.
(b) (4 pts) Find an approximation to this integral, \( T_4 \), using the Trapezoid Rule and \( n = 4 \) subintervals Do not simplify your answer.
(c) (12 pts) Determine reasonable upper bounds on the error in the above approximations. Don’t forget to find bounds for both of the approximations! Hint: \( |x + y| \leq |x| + |y| \)

Solution:

(a) \( \Delta x = \frac{b - a}{n} = \frac{4 - 0}{4} = 1 \) and \( f(x) = e^{-x^2} \), so
\[
M_4 = \Delta x (f(1/2) + f(3/2) + f(5/2) + f(7/2)) = e^{-1/4} + e^{-9/4} + e^{-25/4} + e^{-49/4}
\]

(b) \( \Delta x = 1 \) and \( f(x) = e^{-x^2} \) still, so
\[
T_4 = \frac{\Delta x}{2} (f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)) = \frac{1}{2} (1 + 2e^{-1} + 2e^{-4} + 2e^{-9} + e^{-16})
\]
or \[
\frac{1}{2} + e^{-1} + e^{-4} + e^{-9} + \frac{1}{2} e^{-16}
\]
(c) \( E_{M4} \leq \frac{K(b-a)^3}{24n^2} \) and \( E_{T4} \leq \frac{K(b-a)^3}{12n^2} \), so we need to find \( K \), where \( |f''(x)| \leq K \) for \( 0 \leq x \leq 4 \).

\( f(x) = e^{-x^2} \rightarrow f'(x) = -2xe^{-x^2} \rightarrow f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} \)

\( \rightarrow |f''(x)| = |-2e^{-x^2} + 4x^2e^{-x^2}| \leq |-2e^{-x^2}| + |4x^2e^{-x^2}| \leq 2(e^0) + 4(4^2)(e^0) = 2 + 4(16) = 66 \)

OR

\( \rightarrow |f''(x)| = |-2e^{-x^2} + 4x^2e^{-x^2}| \leq |(0) + 4x^2e^{-x^2}| = 4x^2e^{-x^2} \leq 4(4^2)e^0 = 64 \)

using \( K = 64 \) we have:

\( E_{M4} \leq \frac{K(b-a)^3}{24n^2} = \frac{64(4)^3}{24(4)^2} = \frac{32}{3} \) and \( E_{T4} \leq \frac{K(b-a)^3}{12n^2} = \frac{64(4)^3}{12(4)^2} = \frac{64}{3} = 22 \)

3. Consider the parametric curve defined by \( x = t^2 - 1, y = \sin t, 0 \leq t \leq 2\pi \).

(a) (5 pts) Find an expression for the slope of the tangent line to this curve as a function of \( t \).

(b) (5 pts) What is the equation of the tangent line to the curve at \( t = 0 \)?

(c) (6 pts) At what \( t \) values is the tangent horizontal? Vertical? Remember: \( 0 \leq t \leq 2\pi \)

Solution:

(a) \( \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{2t} \)

(b) \( x = 0 \rightarrow t = \pm 1 \) but \( 0 \leq t \leq 2\pi \), so \( t = 1 \).

Point-slope form of a line: \( y - y_1 = m(x - x_1) \rightarrow y - (\sin 1) = \frac{\cos 1}{2(1)}(x - (0)) \rightarrow y = \frac{\cos 1}{2}x + \sin 1 \)

(c) Horizontal tangent: \( \frac{dy}{dx} = 0 = \frac{\cos t}{2t} \rightarrow \cos t = 0 \rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2} \)

Vertical tangent: \( \frac{dy}{dx} = \pm \infty = \frac{\cos t}{2t} \rightarrow t = 0 \)

4. Consider the polar curve defined by \( r = \frac{\pi}{\theta}, \pi \leq \theta \leq 2\pi \).

(a) (4 pts) Provide a rough sketch of the curve. Indicate direction with an arrow.

(b) (8 pts) Set up but do not evaluate an integral for the length of this curve. Simplify as much as possible.

(c) (8 pts) What is the area between this curve and the circle \( r = 1 \)? Simplify as much as possible.

Solution:

(a) \( r = \frac{\pi}{\theta} \rightarrow \frac{dr}{d\theta} = -\frac{\pi}{\theta^2} \)

\( L = \int_{\pi}^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_{\pi}^{2\pi} \sqrt{\frac{\pi^2}{\theta^2} + \frac{\pi^2}{\theta^4}} d\theta = \int_{\pi}^{2\pi} \pi \sqrt{\frac{\theta^2 + 1}{\theta^4}} d\theta = \int_{\pi}^{2\pi} \frac{\pi}{\theta^2} \sqrt{\theta^2 + 1} d\theta \)
(c) The curve $r = 1/\theta$ is always inside the circle $r = 1$, so the area is

$$A = \int_{\pi}^{2\pi} \frac{1}{2} \left( 1 - \left( \frac{\pi}{\theta} \right)^2 \right) d\theta = \frac{1}{2} \left[ \theta \left( \frac{\pi}{2} \right) + \frac{\pi^2}{2} - \frac{\pi^3}{2} \right]_{\pi}^{2\pi}$$

$$= \frac{1}{2} \left( 2\pi + \frac{\pi^2}{2\pi} - \frac{\pi^2}{\pi} \right) = \frac{1}{2} \left( 2\pi + \frac{\pi}{2} - \pi - \pi \right) = \frac{\pi}{4}$$

5. **Always true or False? (20 pts)**

(a) All intersections between polar curves $r_1 = f(\theta)$ and $r_2 = g(\theta)$ may be found by setting $f(\theta) \equiv g(\theta)$.

(b) The parametric curves $x = t, y = t^2$ and $x = t^2, y = t^4$ have the same graph.

(c) Given the parametric curve $x = f(t), y = g(t),$ $\frac{d^2 y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right)$.

(d) Given the sequence of partial sums, $\{S_n\}$, of a series $\sum a_n$, if $\lim_{n \to \infty} S_n \neq 0$, then $\sum a_n$ is divergent.

**Solution:**

(a) FALSE. That may overlook the pole (origin).

(b) FALSE. $x = t, y = t^2$ gives the parabola $y = x^2$, where $x$ may be positive or negative, and $x = t^2, y = t^4$ gives the same parabola but $x$ can only be positive.

(c) FALSE. $\frac{d^2 y}{dx^2} \neq \frac{d}{dt} \left( \frac{dy}{dx} \right)$.

(d) FALSE. The series may still converge, just to something other than 0. Remember: $\sum a_n = \lim_{n \to \infty} S_n$. 