1. (a) (6 pts) Find the x-coordinate of the center of mass of the infinite region bounded by \( x = 0, \ y = 0 \) and \( y = \lambda e^{-\lambda x} \), where \( \lambda > 0 \). You may use the fact that \( \int_0^\infty \lambda e^{-\lambda x} \, dx = 1 \), for \( \lambda > 0 \).

(b) (4 pts) Write out the full partial fraction decomposition of \( \frac{3x^2 + 1}{x^4 + 3x^3 + 2x^2} \). Do not determine the coefficients.

Solution:

(a) \( \bar{x} = \frac{\sum \int_0^\infty \lambda x e^{-\lambda x} \, dx}{\int_0^\infty \lambda e^{-\lambda x} \, dx} = \lim_{t \to \infty} \int_0^t \lambda x e^{-\lambda x} \, dx \)

integrate by parts with \( u = \lambda x \), \( du = \lambda \, dx \), \( v = -\frac{1}{\lambda} e^{-\lambda x} \), \( dv = e^{-\lambda x} \, dx \) to get:

... = \( \lim_{t \to \infty} \left[ -xe^{-\lambda x} \Bigg|_0^t - \frac{1}{\lambda} \int_0^t e^{-\lambda x} \, dx \right] \)

= \( \lim_{t \to \infty} \left( -\frac{t}{e^\lambda} - \frac{1}{\lambda} e^{-\lambda t} + \frac{1}{\lambda} \right) \)

LH = \( \lim_{t \to \infty} \left( -\frac{1}{\lambda e^\lambda} - \frac{1}{\lambda} e^{-\lambda t} + \frac{1}{\lambda} \right) = \frac{1}{\lambda} \)

(b) \( \frac{3x^2 + 1}{x^4 + 3x^3 + 2x^2} = \frac{3x^2 + 1}{x^2(x^2 + 3x + 2)} = \frac{3x^2 + 1}{x^2(x + 2)(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 2} + \frac{D}{x + 1} \)

2. (a) (4 pts) Suppose the \( n^{th} \) partial sum of the series \( \sum a_n \) is given by \( S_n = n^{1/n} \). Find an expression for the \( n^{th} \) term of the series, \( a_n \). Do not simplify your answer.

(b) (4 pts) Determine whether the series above is convergent or divergent.

c) (6 pts) Is the series \( \sum_{n=100}^\infty \frac{\sin n}{(n^2 + \cos^2 n) \arctan n} \) absolutely convergent, conditionally convergent or divergent? Name any tests you use.

Solution:

(a) \( a_n = S_n - S_{n-1} = n^{1/n} - (n-1)^{1/(n-1)} \)

(b) \( \sum a_n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} n^{1/n} (\text{given}) = 1 \), so \( \boxed{\text{CONVERGENT}} \).

(c) \( 0 \leq \frac{\sin n}{(n^2 + \cos^2 n) \arctan n} \leq \frac{1}{(n^2 + \cos^2 n) \arctan n} \leq \frac{1}{n^2 \arctan n} \)

Therefore, \( \boxed{\text{ABSOLUTELY CONVERGENT BY DIRECT COMPARISON TEST}} \).

3. (a) (4 pts) What function is represented by the power series \( \sum_{n=0}^\infty \frac{x^{n+1}}{n+1} \)? What is the radius of convergence?

(b) (4 pts) Find the sum: \( \sum_{n=0}^\infty \frac{(-1)^n (\pi/2)^{2n+1}}{(2n+1)!} \)

Solution:

(a) \( \frac{1}{1-x} = \sum_{n=0}^\infty x^n \)

\( \int \frac{1}{1-x} \, dx = \int \sum_{n=0}^\infty x^n \, dx \)
\[ -\ln |1 - x| + C \begin{array}{c} |x| < 1 \\ n \end{array} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \]

Plugging in \( x = 0 \) reveals that \(-\ln |1| + C = \sum_{n=0}^{\infty} 0 = 0 \rightarrow C = 0\), leaving as our answer \(-\ln |1 - x|\).

\[ R = 1 \] since the original series has R=1 and all we did was integrate.

(b) \( \sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \), with \( R = \infty \) so \( \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n+1}}{(2n+1)!} = \frac{\sin \pi}{2} = 0 \).  

4. (10 pts) Consider the finite region in the first quadrant bounded by \( y = x \) and \( y = x^2 \).

(a) Set up but do not evaluate integral(s) to find the volume of the solid formed by rotating this region about the line \( x = 1 \) using cylindrical shells.

(b) Same as (a), but using disks/washers.

(c) CHECK to make sure you have the proper method on the proper part above. You will lose points if you have them mixed up!

Solution:

(a) Endpoints: \( x^2 \rightarrow x = 0, 1 \rightarrow \) points: \((0,0), (1,1)\)

\[ dV = 2\pi rhdr = 2\pi (1-x)(x-x^2)dx \rightarrow V = \int_0^1 2\pi (1-x)(x-x^2)dx \]

(b) \[ dV = \pi (R^2 - r^2)dh = \pi ((1-y)^2 - (1 - \sqrt{y})^2) dy \rightarrow V = \int_0^1 \pi ((1-y)^2 - (1 - \sqrt{y})^2) dy \]

5. (8 pts) Consider the curve \( f(x) = \cosh x \) between \( x = 0 \) and \( x = 1 \).

(a) Find the length of this curve.

(b) Set up but do not evaluate an integral for the surface area of the solid formed by rotating the region bounded by this curve, the \( x \)-axis, \( x = 0 \) and \( x = 1 \) about the \( x \)-axis.

Solution:

(a) \[ dL = \sqrt{1 + (f'(x))^2}dx = \sqrt{1 + (\sinh x)^2}dx = \sqrt{\cosh^2 x}dx = \cosh x \, dx \]

\[ L = \int_0^1 \cosh x \, dx = \sinh x \bigg|_0^1 = \sinh 1 \]

(b) \[ dSA = 2\pi rdL = 2\pi (\cosh x)(\cosh x \, dx) \]

\[ SA = \int_0^1 2\pi \cosh^2 x \, dx = \int_0^1 \frac{\pi}{2} (e^{2x} + 2 + e^{-2x}) \, dx = \int_0^1 \pi (1 + \cosh(2x)) \, dx \]

6. (1 pt Extra Credit) Always true or False? If you derive the function \( f(x) = \arctan x \), you get \( \frac{1}{1+x^2} \).

Solution: False! “Derive” and “differentiate” are two very different things.