1. (20 pts) **DO NOT SIMPLIFY YOUR ANSWERS FOR THIS PROBLEM.**

Consider the region bounded by \( y = \sqrt{1 - x^2}, x = e^y, y = -2 \) and the y-axis.

(a) Rotate this region about the line \( x = 1 \). Set up, but **do not evaluate**, an integral(s) for the volume of the resulting solid **using cylindrical shells**.

(b) Rotate this region about the line \( x = 1 \). Set up, but **do not evaluate**, an integral(s) for the volume of the resulting solid **using disks/washers**.

**Solution:**

(a) Note that \( x = e^y \rightarrow y = \ln x \). The region of interest goes from \( x = 0 \) (the y-axis) to the intersection of \( y = \sqrt{1 - x^2} \) and \( y = \ln x \), which you cannot solve analytically; you must recognize that the upper endpoint is \( x = 1 \). But the lower curve (for calculating the height of the cylindrical shells) changes from \( y = -2 \) to \( y = \ln x \) where the two intersect: \(-2 \leq \ln x \leq x = e^{-2}\). Therefore...

For cylindrical shells: \( dV = 2\pi rh \, dr \), where \( dr = dx \), \( r = 1 - x \), \( h = \sqrt{1 - x^2} - (-2) = \sqrt{1 - x^2} + 2 \) for \( 0 \leq x \leq e^{-2} \), and \( h = \sqrt{1-x^2} - \ln x \) for \( e^{-2} \leq x \leq 1 \).

\[
V = \int dV = 2\pi \int_0^{e^{-2}} (1 - x) \left( \sqrt{1-x^2} + 2 \right) \, dx + 2\pi \int_{e^{-2}}^1 (1 - x) \left( \sqrt{1-x^2} - \ln x \right) \, dx
\]

(b) The region is between \( y = -2 \) (given) and \( y = 1 \) (where the circle intersects the y-axis).

For washers: \( dV = \pi (R^2 - r^2) \, dh \). \( dh = dy \) and note that the region must be broken up at \( y = 0 \) since the inner radius changes from \( r = 1 - (e^y) \) to \( r = 1 - \sqrt{1-y^2} \). The outer radius is always \( R = 1 \).

\[
V = \pi \int_{-2}^0 (1 - (1 - e^y)^2) \, dy + \pi \int_0^1 (1 - \sqrt{1-y^2})^2 \, dy
\]

\[
= \pi \int_{-2}^0 \left( 2e^y - e^{2y} \right) \, dy + \pi \int_0^1 \left( y^2 - 1 + 2\sqrt{1-y^2} \right) \, dy
\]

The region of interest is shown below.

![Region of Interest](image)

2. (16 pts) Consider the curve \( f(x) = \ln(\cos x), 0 \leq x \leq \frac{\pi}{3} \).

(a) Set up but **do not evaluate** the integral(s) necessary to find the arc length of this curve between these points.

(b) Set up but **do not evaluate** the integral(s) necessary to find the surface area of the solid resulting from rotating this curve about the x-axis. **Use vertical partitioning strips**.

**Solution:**

(a) \( f(x) = \ln(\cos x) \rightarrow f'(x) = \frac{1}{\cos x} (-\sin x) = -\tan x \rightarrow 1 + (f'(x))^2 = 1 + (-\tan x)^2 = \sec^2 x \), giving...
\[ L = \int_0^\frac{\pi}{2} \sqrt{1 + (f'(x))^2} \, dx = \int_0^\frac{\pi}{2} \sqrt{\sec^2 x} \, dx = \int_0^\frac{\pi}{2} \sec x \, dx \]

(b) \( dA = 2\pi r dL \). Noting from the reminder given in the problem that \( f(x) < 0 \) for \( 0 \leq x \leq \frac{\pi}{3} \), \( r = -f(x) = -\ln(\cos x) \). So:

\[ A = 2\pi \int_0^\frac{\pi}{2} (-\ln(\cos x)) \sec x \, dx \]

3. (14 pts) These nifty wine bottle holders use the concept of center-of-mass to hover the bottle in mid-air. Approximate the holder as the region between the lines \( y = 2 \), \( y = -x \), \( y = 0 \) and \( y = -x - \frac{1}{4} \), which is the dashed red parallelogram in the figure. Approximate the wine bottle as the rectangular region given by \( 1 \leq y \leq 2 \) and \(-1 \leq x \leq 2 \). Let \( \rho_1 \) denote the uniform density of the holder and \( \rho_2 \) denote the uniform density of the wine bottle. As you might guess, the contraption will fall over if the x-coordinate of its center of mass is not within the base of the holder. Therefore, find an expression for \( \rho_1 \) in terms of \( \rho_2 \) which will make the x-coordinate of the system’s center of mass be \( \bar{x} = -\frac{1}{8} \).

**Solution:** The idea: \( \bar{x} = \frac{m_1 \bar{x}_1 + m_2 \bar{x}_2}{m_1 + m_2} \), where \( m_1 \) and \( m_2 \) are the masses of the holder and wine bottle, respectively, and \( \bar{x}_1 \) and \( \bar{x}_2 \) are the x-coordinates of the centers of mass of the holder and wine bottle.

From symmetry, we can see that \( \bar{x}_1 = \frac{1}{2} * \frac{-9}{4} = -\frac{9}{8} \) and \( \bar{x}_2 = \frac{1}{2} * (2 - 1) = \frac{1}{2} \).

Now the masses are \( m_1 = \rho_1 A_1 \) and \( m_2 = \rho_2 A_2 = 3\rho_2 \) since the wine bottle is just a rectangle.

You may find the area of the holder in one of two ways:

1. The area of a sheared rectangle (parallelogram) is the same as the original rectangle. Imagine grabbing the top of the parallelogram and sliding over so the right side of it is flush with the y-axis. This creates a normal rectangle of area \( A_1 = \frac{1}{2} \times 2 = \frac{1}{2} \).

2. The holder is the sum of 2 equal area triangles and the region between the lines \( y = -x \) and \( y = -x - \frac{1}{4} \) for \(-2 \leq x \leq -\frac{1}{4} \).

The triangles have base and height both equal to \( \frac{1}{4} \) since the lines are shifted by \( \frac{1}{4} \) from one another. Thus the area of each triangle is \( \frac{1}{2} \left( \frac{1}{4} \right) \left( \frac{1}{4} \right) = \frac{1}{32} \).

Between the lines, we have \( \int_{-2}^{-\frac{1}{4}} ((-x) - (-x - \frac{1}{4})) \, dx = \int_{-2}^{-\frac{1}{4}} \frac{1}{4} \, dx = \frac{1}{4} \left( -\frac{1}{4} - -2 \right) = \frac{17}{16} = \frac{7}{8} \).

Thus \( A_1 = 2 \frac{17}{32} + \frac{7}{16} = \frac{8}{16} = \frac{1}{2} \).

Now, plugging everything into the equation for \( \bar{x} = -\frac{1}{8} \) gives:

\[ -\frac{1}{8} \equiv \frac{\rho_1 (\frac{1}{2}) (\frac{-9}{8}) + \rho_2 (3)(\frac{1}{2})}{\rho_1 (\frac{1}{2}) + \rho_2 (3)} \]

\[ \rightarrow -\frac{1}{16} \rho_1 - \frac{3}{8} \rho_2 \equiv -\frac{9}{16} \rho_1 + \frac{3}{2} \rho_2 \rightarrow \frac{8}{16} \rho_1 \equiv \frac{15}{8} \rho_2 \rightarrow \rho_1 \equiv \frac{15}{4} \rho_2 \]

4. (a) (8 pts) Suppose the volume of Kool-Aid at your party may be modeled as \( \frac{dV}{dt} = V(-1 + \cos t) \), where \( V \) is the volume of Kool-Aid at any given instant in time and \( t \) is time. If you and your roommates initially bought 90 Kool-Aids, find an equation for your supply of Kool-Aid as a function of time.

(b) (7 pts) Suppose, instead, you model your dwindling supply of Kool-Aid as \( \frac{dV}{dt} = 3t^2 - 18t \). Using the same initial condition as in (a), find a new equation for your supply of Kool-Aid as a function of time.

(c) (0 pts) Take a second to note that this is awesome! Now you can figure out whether your initial supply of Kool-Aid will be sufficient, or if you’ll be making a Kool-Aid run later.
Solution: (a) separation of variables: \( \frac{dV}{dt} = V(-1 + \cos t) \rightarrow \frac{dV}{V} = (-1 + \cos t) \ dt \rightarrow \int \frac{dV}{V} = \int (-1 + \cos t) \ dt \rightarrow \ln |V| = -t + \sin t + C \rightarrow V(t) = e^{-t+\sin t}e^C = Ae^{-t+\sin t} \)

Apply the initial condition, that \( V(0) = 90 \), to find \( V(0) = 90 = Ae^{0+\sin 0} = A \rightarrow A = 90 \). The answer is then \( V(t) = 90e^{-t+\sin t} \).

(b) separation of variables again: \( \frac{dV}{dt} = 3t^2 - 18t \rightarrow dV = (3t^2 - 18t)dt \rightarrow \int dV = \int (3t^2 - 18t)dt \rightarrow V(t) = t^3 - 9t^2 + C \)

apply the initial condition: \( V(0) = 90 = (0) - (0) + C \rightarrow C = 90 \). The answer is then \( V(t) = t^3 - 9t^2 + 90 \).

(c) This sure is neat! Check out what the modeled Kool-Aid supply looks like for both of your potential models:

\[ V(t) = t^3 - 9t^2 + 90 \]

5. (20 pts) Determine whether the following sequences converge or diverge. For those that converge, find the limit. Justify your answers, show all work:

(a) \( \left\{ \frac{n-1}{(n^2 + \sin^2 n) \arctan(n)} \right\}_{n=100}^{\infty} \)

(b) \( \left\{ (1 + n)^{\frac{1}{n}} \right\}_{n=1}^{\infty} \)

Solution: Put the Squeeze Theorem to work on this thing:

Since \( n \) starts at 100, we know that the numerator is certainly positive. Also: For \( 100 \leq n < \infty \), \( \arctan n \geq \arctan(100) > 0 \) and \( n^2 + \sin^2 n > 0 \). Therefore:

\[ 0 \leq \frac{n-1}{(n^2 + \sin^2 n) \arctan(n)} \leq \frac{n}{(n^2 + \sin^2 n) \arctan(100)} \leq \frac{n}{n^2 \arctan(100)} = \frac{1}{n \arctan(100)} \]

and \( \lim_{n \to \infty} \frac{1}{n \arctan(100)} = 0 \)

therefore, \( \lim_{n \to \infty} \frac{n-1}{(n^2 + \sin^2 n) \arctan(n)} = 0 \) by the Squeeze Theorem.

(b) \( a_n = (1 + n)^{\frac{1}{n}} \rightarrow \ln(a_n) = \frac{1}{n} \ln(1 + n) \), which is an indeterminant limit as \( n \to \infty \). Use L’Hopital’s rule:

\[ \lim_{n \to \infty} \frac{\ln(1 + n)}{n} = \lim_{n \to \infty} \frac{1}{1 + n} = 0 \]

\[ \rightarrow \lim_{n \to \infty} \ln(a_n) = 0 \rightarrow \lim_{n \to \infty} a_n = e^0 = 1 \]
6. (15 pts) Answer either **Always True** or **False**. Do **NOT** justify your answer.

(a) If \( \lim_{n \to \infty} a_n = 2 \), then \( \lim_{n \to \infty} (a_{2n+1} - a_{2n}) = 0 \).

(b) If \( \{a_n\} \) and \( \{b_n\} \) are divergent sequences, then the sequence \( \{a_n + b_n\} \) is divergent.

(c) \( T(t) = e^{-kt} \) is a solution to the differential equation \( \frac{dT}{dt} + k(T - A) = 0 \), where \( k \) and \( A \) are positive constants.

**Solution:**
(a) **TRUE:** since \( a_n \) is convergent, all sub-sequences must also converge to the same limit, and you can distribute the limit into the parentheses. So

\[
\lim_{n \to \infty} (a_{2n+1} - a_{2n}) = \lim_{n \to \infty} a_{2n+1} - \lim_{n \to \infty} a_{2n} = 2 - 2 = 0.
\]

(b) **FALSE:** Counterexample: \( a_n = (-1)^n \) and \( b_n = (-1)^{n+1} \) are both divergent, but adding them together yields \( a_n + b_n = 0 \) for all \( n \), which is a convergent sequence.

(c) **FALSE:** take the derivative and plug it into the differential equation to see if the alleged solution works:

\[
\frac{dT}{dt} = -ke^{-kt} \rightarrow \frac{dT}{dt} + k(T - A) = (-ke^{-kt}) + k((e^{-kt}) - A) = -kA \neq 0
\]