Solution: APPM 1360

1. (12 pts ea.) Evaluate the integrals, show all work:

(a) \[ \int \frac{x}{(\sqrt{x^2+1})^3(\sqrt{x^2-1})^3} \, dx \]
(b) \[ \int_0^\frac{\pi}{2} \frac{\tan \theta \sec^2 \theta}{\tan^2 \theta + 3 \tan \theta + 2} \, d\theta \]
(c) \[ \int \ln x \, dx \]

Solution:

(a) \[ \int \frac{x}{(\sqrt{x^2+1})^3(\sqrt{x^2-1})^3} \, dx = \int \frac{x}{(x^2 - 1)^{3/2}} \, dx \]

let \( u = x^2 \), \( du = 2xdx \), to get

\[ \cdots = \frac{1}{2} \int \frac{du}{(u^2 - 1)^{3/2}} \]

then let \( u = \sec \theta \), \( du = \sec \theta \tan \theta \, d\theta \) to get

\[ \cdots = \frac{1}{2} \int \sec \theta \tan \theta \sec^2 \theta \, d\theta = \frac{1}{2} \int \sec \theta \tan \theta \, d\theta = \frac{1}{2} \int \frac{1}{\tan^2 \theta} \, d\theta = \frac{1}{2} \int \frac{1}{\sin^2 \theta} \, d\theta = \frac{1}{2} \int \cos \theta \, d\theta \]

let \( v = \sin \theta \), \( dv = \cos \theta \, d\theta \), to get

\[ \cdots = \frac{1}{2} \int \frac{dv}{v^2} = -\frac{1}{2v} + C = -\frac{1}{2 \sin \theta} + C \]

setting up the triangle from \( u = \sec \theta \) (hypoteneuse=\( u \), adjacent side=1, opposite side=\( \sqrt{u^2 - 1} \)) shows that \( \sin \theta = \frac{\sqrt{u^2 - 1}}{u} \), giving

\[ \cdots = -\frac{1}{2} \left( \frac{1}{\sqrt{u^2 - 1}} \right) + C = -\frac{u}{2u^2 - 1} + C = -\frac{x^2}{2x^2 - 1} + C \]

(b) Let \( u = \tan \theta \), \( du = \sec^2 \theta \, d\theta \) to get

\[ \int_0^{\frac{\pi}{2}} \frac{\tan \theta \sec^2 \theta}{\tan^2 \theta + 3 \tan \theta + 2} \, d\theta = \int_0^1 \frac{u}{u^2 + 3u + 2} \, du = \int_0^1 \frac{u}{(u+2)(u+1)} \, du \]

now use partial fractions:

\[ \frac{u}{(u+2)(u+1)} = \frac{A}{u+2} + \frac{B}{u+1} \]

\[ \rightarrow u = A(u+1) + B(u+2) \] (from multiplying both sides by the left side’s denominator)

let \( u = -1 \). then \(-1 = A(0) + B(1) \rightarrow B = -1 \)

let \( u = -2 \). then \(-2 = A(-1) + B(0) \rightarrow A = 2 \). therefore...

\[ \cdots = \int_0^1 \left( \frac{2}{u^2} - \frac{1}{u+1} \right) \, du = (2 \ln(u+2) - \ln(u+1)) \big|_0^1 = ((2 \ln(3) - \ln(2)) - (2 \ln(2) - \ln(1))) = 2 \ln 3 - 3 \ln 2 = \ln 9 - \ln 8 = \ln \frac{9}{8} \]

(c) integrate by parts: \( u = \ln x \), \( du = \frac{1}{x} \, dx \); \( v = x \), \( dv = dx \) gives

\[ \cdots = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx = x \ln x - x + C \]
3. (10 pts) If you wanted to find the area of an ellipse, which comes up in many astrophysical applications, you would probably find yourself evaluating the following integral. Make Future You proud and do the integral now.

\[ \int_0^a \sqrt{1 - \left(\frac{x}{a}\right)^2} \, dx \]

**Solution:** Begin with the u-sub \( u = \frac{x}{a}, \, du = \frac{1}{a} \, dx \) to get

\[ b \int_0^a \sqrt{1 - \left(\frac{x}{a}\right)^2} \, dx = ab \int_0^1 \sqrt{1 - u^2} \, du. \]

It is perfectly acceptable to recognize this as the area of one-fourth of a circle of radius 1, which gives you that the integral there is equal to \( \frac{\pi}{4} \), and thus the answer is \( \frac{\pi}{4} ab \). But if you didn’t see that...

Use the trig sub: \( u = \sin \theta, \, du = \cos \theta \, d\theta \). Thus your integral becomes:

\[ I = ab \int_0^\frac{\pi}{2} \sqrt{1 - \sin^2 \theta} \cos \theta \, d\theta = ab \int_0^\frac{\pi}{2} \cos^2 \theta \, d\theta = ab \int_0^\frac{\pi}{2} \frac{1}{2} (1 + \cos 2\theta) \, d\theta = \frac{ab}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) \bigg|_0^{\frac{\pi}{2}} = \frac{\pi ab}{4} \]
4. (15 pts) Boulder Reservoir may be VERY roughly parameterized as the area enclosed by \( y = x + 3, y = x^2 - 2x + 3, \ y = -2x + 7, y = x - 2, y = 0 \) and \( y = -2x \), as shown in the picture. Find the surface area of Boulder Reservoir. Express your final answer as a single fraction.

Note: A larger version of this picture is on the other side with the formulas.

**Solution:** The area is really 3 areas: A1 bounded above by \( y = x + 3 \) and below by \( y = -2x \), A2 bounded above by \( y = x^2 - 2x + 3 \) and below by \( y = 0 \), A3 bounded above by \( y = -2x + 7 \) and below by \( y = x - 2 \). So your answer will be the sum of these 3 areas, each their own integral.

Find the bounds for each area integral by finding the relevant intersections. A1 starts where \( y = x + 3 \) and \( y = -2x \) intersect: \( x + 3 = -2x \to 3x + 3 = 0 \to x = -1 \)

A1 ends where \( y = x + 3 \) and \( y = x^2 - 2x + 3 \) intersect: \( x + 3 = x^2 - 2x + 3 \to x^2 - 3x = 0 \to x(x-3) = 0 \to x = 0 \) (since we clearly don’t want the \( x = 3 \) root). This also corresponds to where the bottom curve in A1, \( y = -2x \), hits the x-axis, so we do not need to break up A1 into 2 mini-integrals with different bottom curves.

A2 starts where A1 ended (\( x = 0 \)) and ends where \( y = x^2 - 2x + 3 \) and \( y = -2x + 7 \) intersect: \( x^2 - 2x + 3 = -2x + 7 \to x^2 - 4 = 0 \to x = \pm 2 \to x = 2 \), since clearly we want the positive root. This also corresponds to where the bottom curve (\( y = 0 \)) intersects the next bottom curve (\( y = x - 2 \)), so we do not need to separate A2 into 2 mini-integrals with different bottom curves either.

A3 starts where A2 ended (\( x = 2 \)) and ends where \( y = -2x + 7 \) and \( y = x - 2 \) intersect: \( -2x + 7 = x - 2 \to -3x + 9 = 0 \to x = 3 \)

Therefore, we have:

\[
A1 = \int_{-1}^{0} ((x + 3) - (-2x)) \, dx = \int_{-1}^{0} (3x + 3) \, dx = \left[ \frac{3}{2} x^2 + 3x \right]_{-1}^{0} = -\left( \frac{3}{2} - 3 \right) = \frac{3}{2}
\]

\[
A2 = \int_{0}^{2} ((x^2 - 2x + 3) - (0)) \, dx = \left[ \frac{1}{3} x^3 - x^2 + 3x \right]_{0}^{2} = \frac{1}{3} (8) - (4) + (6) = \frac{8}{3} + 2 = \frac{14}{3}
\]

\[
A3 = \int_{2}^{3} ((-2x + 7) - (x - 2)) \, dx = \int_{2}^{3} (-3x+9) \, dx = \left[ -\frac{3}{2} x^2 + 9x \right]_{2}^{3} = \left( -\frac{3}{2} (9) + 27 \right) - \left( -\frac{3}{2} (4) + 18 \right) = \frac{27}{2} - 12 = \frac{3}{2}
\]

Thus: \( A = A1 + A2 + A3 = \frac{3}{2} + \frac{14}{3} + \frac{3}{2} = 3 + \frac{14}{3} = \frac{23}{3} \)

5. (5 pts ea.) Mystery Bag (it could be anything!)

(a) **True or false? Show no work:** If \( f(x) \) is continuous, then \( \int_{-\infty}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{-t}^{t} f(x) \, dx \).

(b) **True or false? Show no work:** You could have used a Comparison Test with \( f(x) = x^2 \) in 2(a) to get the same answer.

(c) Can you factor \( x^3 + 1 \) into distinct linear and irreducible quadratic terms? If you can, do it.

**Solution:**

(a) False. must split up into two limits: \( \int_{-\infty}^{\infty} f(x) \, dx = \lim_{t \to -\infty} \int_{t}^{0} f(x) \, dx + \lim_{s \to \infty} \int_{0}^{s} f(x) \, dx \)
(b) False. $x^2$ is positive, but $x^2 \ln x$ is negative on $[0,1]$, so no comparison can be made between them.

(c) Yes. It has been known for a LONG time that you can always factor any polynomial of degree 3 or greater into linear and quadratic terms. Use polynomial long division to find $x^3 + 1 = (x+1)(x^2 - x + 1)$. Polynomial long division is demonstrated in section 6.6 of Essential Calculus.