1. (28 points) State whether each of the following quantities converge or diverge. Explain your reasoning.

(a) The sequence \( a_1, a_2, a_3, \ldots \) where \( a_n = \ln(8n) - \ln(n + 1) \)

(b) \( \int_0^2 x \ln x \, dx \)

(c) \( \sum_{n=1}^{\infty} \frac{1}{(\arctan(n))^{1/n}} \)

(d) \( \sum_{n=1}^{\infty} \frac{(n!)^2}{e^{n^2}} \)

**Solution:**

(a) The sequence **converges** to \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{8n}{n + 1} = L H \ln 8. \)

(b) The integral is **convergent**:

\[
\int_0^2 x \ln x \, dx = \lim_{t \to 0^+} \int_t^2 x \ln x \, dx
\]

\[
= \lim_{t \to 0^+} \left( \left[ \frac{x^2}{2} \ln x \right]_t^2 - \int_t^2 \frac{x}{2} \, dx \right)
\]

\[
= \lim_{t \to 0^+} \left( 2 \ln 2 - 1 - \left( \frac{t^2}{2} \ln t - \frac{t^2}{4} \right) \right) = 2 \ln 2 - 1.
\]

Note that

\[
\lim_{t \to 0^+} t^2 \ln t = \lim_{t \to 0^+} \frac{\ln t}{t^{-2}} \overset{L H}{=} \lim_{t \to 0^+} \frac{1/t}{-2t^{-3}} = \lim_{t \to 0^+} -\frac{t^2}{2} = 0.
\]

(c) The series **diverges** by the Test for Divergence:

\[
\lim_{n \to \infty} \frac{1}{(\arctan(n))^{1/n}} = \frac{1}{(\pi/2)^0} = 1 \neq 0.
\]

(d) Apply the Ratio Test.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n + 1)!^2}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{(n!)^2} \right| = \lim_{n \to \infty} \frac{(n + 1)^2}{e^{2n+1}}
\]

\[
= L H \lim_{n \to \infty} \frac{2(n + 1)}{2e^{2n+1}} = L H \lim_{n \to \infty} \frac{2}{4e^{2n+1}} = 0 < 1.
\]

The series absolutely **converges**.
2. (18 points) Consider the circles shown. The largest circle has a radius of 2 m, the next largest circle has a radius of 1 m, and the next has a radius of $\frac{1}{2}$ m, etc. Each circle has half the radius of the previous one. Suppose there is an infinite number of circles and $S$ represents the total area of all the circles.

(a) Express $S$ as an infinite series $\sum_{n=1}^{\infty} a_n$ by finding an expression for $a_n$, where $a_n$ is the area of the $n^{th}$ circle.

(b) Is $a_n$ bounded? If so, find the bounds. If not, explain why not.

(c) Let $s_n$ equal the $n$th partial sum of the series. Find $s_3$. Simplify your answer.

(d) Evaluate $\lim_{n \to \infty} \frac{s_{n+1} - s_n}{\pi}$.

(e) Find the total area $S$ or explain why it does not exist.

Solution:

(a) The series is geometric with $a = 4\pi$ and ratio $r = \frac{1}{4}$.

$$S = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} 4\pi \left(\frac{1}{4}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{\pi}{4^{n-2}}$$

(b) Since $a_n$ is a positive, decreasing sequence, it is bounded above by $a_1 = 4\pi$ and bounded below by $0$, and therefore $a_n$ is bounded.

(c) $s_3 = a_1 + a_2 + a_3 = 4\pi + \pi + \frac{1}{4}\pi = \frac{21}{4}\pi$.

(d) $\lim_{n \to \infty} \frac{s_{n+1} - s_n}{\pi} = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{\pi}{4^{n-1}} = 0$.

Alternate solution: This is a convergent geometric series ($|r| < 1$).

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} s_{n+1} = S \Rightarrow \lim_{n \to \infty} (s_{n+1} - s_n) = S - S = 0.$$  

(e) The sum of the geometric series is $S = \frac{4\pi}{1 - 1/4} = \frac{16}{3}\pi$.

3. (18 points) Let $y = \frac{1}{x^2 - 3x + 2}$.

(a) Consider the shaded region shown bounded by the curve and the $x$-axis, $0 \leq x \leq \frac{1}{2}$. Find the volume of the solid generated by rotating the region about the $y$-axis. Simplify your answer.

(b) Set up but do not evaluate an integral to find the area of the surface generated when the curve, $0 \leq x \leq \frac{1}{2}$, is rotated about the line $x = 1$.

Solution:
(a) By the shell method and partial fraction decomposition

\[
V = \int_0^{1/2} 2\pi \frac{x}{x^2 - 3x + 2} \, dx = \int_0^{1/2} 2\pi \left( \frac{2}{x - 2} - \frac{1}{x - 1} \right) \, dx
\]

\[
= 2\pi \left[ 2 \ln |x - 2| - \ln |x - 1| \right]_0^{1/2} = 2\pi \left( 2 \ln \frac{3}{2} - \ln \frac{1}{2} - 2 \ln 2 + 0 \right)
\]

\[
= \frac{2}{3} \ln \frac{9}{8} \pi
\]

Partial fraction decomposition:

\[
\frac{x}{x^2 - 3x + 2} = \frac{A}{x - 2} + \frac{B}{x - 1} = \frac{A(x - 1) + B(x - 2)}{x^2 - 3x + 2}
\]

\[
x = A(x - 1) + B(x - 2)
\]

Let \( x = 1 \). Then \( B = -1 \).

Let \( x = 2 \). Then \( A = 2 \).

\[
\frac{x}{x^2 - 3x + 2} = \frac{2}{x - 2} + \frac{-1}{x - 1}
\]

Alternatively:

Match terms: \((A + B)x = x\) and \(-A - 2B = 0\), so \( A = 2 \) and \( B = -1 \).

(b) \( S = \int_0^{1/2} 2\pi (1 - x) \sqrt{1 + \left( \frac{-2x - 3}{x^2 - 3x + 2} \right)^2} \, dx \)

4. (20 points) The function \( g(x) \) equals the Taylor series

\[
g(x) = \frac{x^2}{2} - \frac{x^3}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4} - \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} + \cdots
\]

(a) Use the fourth degree Taylor polynomial \( T_4(x) \) to approximate the value of \( g(1) \). Simplify your answer.

(b) Find an error bound for the approximation in part (a). Justify your answer.

(c) What is the exact value of \( g(1) \)?

(d) Find a series representation for \( g'(x) \). Express your answer in sigma notation starting with \( n = 1 \). Simplify your answer.

Solution:

(a)

\[
T_4(x) = \frac{x^2}{2} - \frac{x^3}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4}
\]

\[
g(1) \approx T_4(1) = \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{9}{24} = \frac{3}{8}
\]
(b) The series \( g(1) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \) satisfies the conditions of the Alternating Series Test with \( b_n = 1/n! : \)

\[
\frac{1}{(n+1)!} < \frac{1}{n!} \Rightarrow b_{n+1} < b_n \quad \text{and} \quad \lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n!} = 0.
\]

By the Alternating Series Estimation Theorem, an error bound is

\[
|g(1) - T_4(1)| < b_5 = \frac{1}{5!} = \frac{1}{120}.
\]

Alternate solution:

Note that the given series \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \) is the Maclaurin series for \( g(x) = e^{-x} + x - 1 \). Using Taylor’s Formula the error is

\[
R_4(x) = \frac{g^{(5)}(z)}{5!} x^5 \quad \text{for} \quad x = 1 \quad \text{and} \quad 0 < z < 1.
\]

The derivative \( g^{(5)}(z) = \left| \frac{-1}{e^z} \right| < \frac{1}{e^0} = 1 \), so \( |R_4(x)| < \frac{1}{5!} \).

(c) The given series \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \) is the Maclaurin series for \( g(x) = e^{-x} + x - 1 \). At \( x = 1 \),

\( g(1) = e^{-1} \).

(d) Differentiating term by term we get

\[
g'(x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n!}.
\]

Alternate Solution: \( g'(x) = \frac{d}{dx} \left( \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n!} \right) = \sum_{n=2}^{\infty} (-1)^n \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n!} \).

5. (26 points) Consider the curve \( C \) given by the parametric equations \( x = a \sin(t) \), \( y = b \cos(t) \), where \( 0 \leq t \leq 2\pi \) and \( a, b \) are positive constants with \( a > b \).

(a) Eliminate the parameter to obtain an equation in rectangular coordinates that describes \( C \).

(b) Sketch a graph of the curve \( C \). In your graph: Indicate the starting and ending positions of the parametric equation. Use an arrow to show the direction the curve is traversed. Label all intercepts.

(c) Using the given parametric equations set up an integral to find the area of the region enclosed by the curve \( C \). Evaluate this integral.

(d) Set up but do not evaluate an integral that equals the arc length of the curve \( C \).

(e) Find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) in terms of the parameter \( t \).

Solution:

(a) Using the identity \( \sin^2 t + \cos^2 t = 1 \), a Cartesian equation for the ellipse is

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
\]
(b)\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

(c) By symmetry
\[
A = 4 \int_0^{\pi/2} b \cos t (a \cos t) \, dt = 4ab \int_0^{\pi/2} \cos^2 t \, dt = 4ab \int_0^{\pi/2} \frac{1}{2} (1 + \cos(2t)) \, dt = 4ab \cdot \frac{1}{2} \left[ t + \frac{1}{2} \sin(2t) \right]_0^{\pi/2} = 2ab \left( \frac{\pi}{2} \right) = ab\pi.
\]

(d) \[
L = \int_0^{2\pi} \sqrt{(a \cos t)^2 + (-b \sin t)^2} \, dt \quad \text{or using the Cartesian form } \quad y = \frac{b}{a} \sqrt{a^2 - x^2},
\]

\[
L = 2 \int_{-a}^{a} \sqrt{1 + \left( \frac{-bx}{a\sqrt{a^2 - x^2}} \right)^2} \, dx.
\]

(e) \[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-b \sin t}{a \cos t} = -\frac{b}{a} \tan t
\]
\[
\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{(b/a) \sec^2 t}{a \cos t} = -\frac{b}{a^2} \sec^3 t = -\frac{b}{a^2 \cos^3 t}.
\]

6. (18 points, 6 points each) Unrelated short answer questions.

(a) Always True or False? If \( f(x) \leq g(x) \) for all \( x \geq 1 \) and \( \int_1^\infty g(x) \, dx \) converges, then \( \int_1^\infty f(x) \, dx \) converges.
(b) Find the \( x^{50} \) term of the Taylor series for \( \sin x \) centered at \( a = \frac{\pi}{2} \).
(c) How large should \( n \) be to guarantee that the Midpoint Rule approximation for \( \int_1^2 x \ln x \, dx \) is accurate to within \( \frac{1}{(4!)^3} \)? Simplify your answer.

Solution:

(a) \underline{False}. For example, let \( f(x) = -\frac{1}{x} \) and \( g(x) = \frac{1}{x^2} \). Then \( f(x) \leq g(x) \) for \( x \geq 1 \) and \( \int_1^\infty \frac{1}{x^2} \, dx \) converges, but \( \int_1^\infty \left( -\frac{1}{x} \right) \, dx \) diverges.

Note that the integral Comparison Theorem applies to functions \( f \) and \( g \) that are both positive.
(b) The $x^{50}$ term of a Taylor series centered at $a$ is $f^{(50)}(a)/50! (x-a)^{50}$. The derivatives for $f(x) = \sin x$ occur in a cycle of 4:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(\pi/2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\sin x$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\cos x$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$-\sin x$</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>$-\cos x$</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$\sin x$</td>
<td>1</td>
</tr>
</tbody>
</table>

It follows that $f^{(50)}(\pi/2) = f^{(2)}(\pi/2) = -1$ and the desired term is $-\frac{1}{50!} (x - \frac{\pi}{2})^{50}$.

(c) Let $f(x) = x \ln x$, $f'(x) = 1 + \ln x$, and $f''(x) = 1/x$. In the interval $[1, 2]$, $|f''| \leq 1$. Solve $|E_M| \leq 1/(4!)^3$ for $n$, letting the upper bound $K = 1$.

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \leq \frac{1}{(4!)^3}$$

$$\frac{1}{24n^2} \leq \frac{1}{24^3} \Rightarrow n \geq 24 \text{ subintervals}$$

7. (22 points) Consider the rose curve $r = \sin(3\theta)$.

(a) Sketch a graph of the curve in the $xy$-plane and the corresponding graph in the $r\theta$-plane. For both graphs, label all intercepts.

(b) Find an equation of the tangent line at the “tip of the leaf” in the first quadrant. [Note: the tip of the leaf occurs when $|r|$ is a maximum].

(c) Show that the tangent line in part (b) is perpendicular to the line that connects the origin to the tip of the leaf.

(d) Set up, but do not evaluate, an integral to find the area inside one leaf of the given curve.

(e) Set up but do not evaluate an integral that equals the exact arc length of one leaf of the given curve.

**Solution:**

(a) 

\[ r = \sin(3\theta) \]
(b) First find the tangent slope. At $\theta = \pi/6$, $|r|$ attains a maximum value of 1.

$$
\frac{dy}{dx} = \frac{d}{d\theta} \frac{(\sin(3\theta) \sin \theta)}{(\sin(3\theta) \cos \theta)} = \frac{\sin(3\theta) \cos \theta + 3 \sin \theta \cos(3\theta)}{-\sin(3\theta) \sin \theta + 3 \cos \theta \cos(3\theta)}
$$

$$
\left. \frac{dy}{dx} \right|_{\theta = \pi/6} = \frac{(1)\sqrt{3}/2 + 0}{-(1)\sqrt{3}/2 + 0} = -\sqrt{3}
$$

The point of tangency is $(r \cos(\pi/6), r \sin(\pi/6)) = \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right)$.

The tangent line is

$$
y = \frac{1}{2} - \sqrt{3} \left( x - \frac{\sqrt{3}}{2} \right)
$$

(c) The line connecting the origin to the point $\left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right)$ has slope $\frac{1/2 - 0}{\sqrt{3}/2 - 0} = \frac{1}{\sqrt{3}}$, which is the negative reciprocal of the tangent slope $-\sqrt{3}$. The lines are therefore perpendicular.

(d) $A = \int_{0}^{\pi/3} \frac{1}{2} \sin^2(3\theta) \, d\theta$

(e) $L = \int_{0}^{\pi/3} \sqrt{\sin^2(3\theta) + (3 \cos(3\theta))^2} \, d\theta$