1. (20 pts, 10 pts ea.) Find the antiderivative: \( a) \int \frac{\ln(x)}{\sqrt{x}} \, dx \quad \text{and} \quad b) \int \frac{2x^2 + x + 1}{x^3 + x^2 + x + 1} \, dx \)

**Solution:**

(a) Using integration by parts with \( u = \ln(x) \Rightarrow du = 1/x \) and \( dv = x^{-1/2} \, dx \Rightarrow v = 2x^{1/2} \) and so,
\[
\int \frac{\ln(x)}{\sqrt{x}} \, dx = 2x^{1/2} \ln(x) - \int \frac{2}{x^{1/2}} \, dx = 2x^{1/2} \ln(x) - 4x^{1/2} + C = 2\sqrt{x} \left[ \ln(x) - 2 \right] + C
\]

(b) Note that \( x^3 + x^2 + x + 1 = x^3 + x + 1 + (x + 1)^2 + 1 \), and so using partial fractions we have
\[
\frac{2x^2 + x + 1}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \Rightarrow 2x^2 + x + 1 = A(x^2 + 1) + (Bx + C)(x + 1)
\]
and so,
\[
A + B = 2 \\
B + C = 1 \\
A + C = 1
\]

thus,
\[
\int \frac{2x^2 + x + 1}{x^3 + x^2 + x + 1} \, dx = \int \frac{1}{x + 1} \, dx + \int \frac{x}{x^2 + 1} \, dx = \ln|x + 1| + \frac{1}{2} \ln|x^2 + 1| + C
\]

2. (20 pts, 10 pts ea.) Determine if the following integrals are convergent or divergent. Justify your answer.

(a) \( \int_{106}^{\infty} \frac{1 + \sin^4(\sqrt{x})}{2\sqrt{x} - 3} \, dx \), \quad \text{and} \quad (b) \int_{0}^{\infty} 2x^3 e^{1+x^2} \, dx

**Solution:**

(a) Note that \( \sin^4(\sqrt{x}) \geq 0 \) for \( x > 0 \) and so,
\[
\frac{1 + \sin^4(\sqrt{x})}{2\sqrt{x} - 3} \geq \frac{1}{2\sqrt{x} - 3} \geq \frac{1}{2\sqrt{x}} \Rightarrow \int_{106}^{\infty} \frac{1 + \sin^4(\sqrt{x})}{2\sqrt{x} - 3} \, dx \geq \int_{106}^{\infty} \frac{1}{2\sqrt{x}} \, dx
\]
where the last inequality follows from the fact that \( 2\sqrt{x} - 3 < 2\sqrt{x} \), and note that
\[
\int_{106}^{\infty} \frac{1}{2\sqrt{x}} \, dx = \lim_{t \to \infty} \int_{106}^{t} \frac{1}{2x^{1/2}} \, dx = \lim_{t \to \infty} x^{1/2} \bigg|_{106}^{t} = \lim_{t \to \infty} \sqrt{t} - \sqrt{106} = +\infty
\]
and so \( \int_{106}^{\infty} \frac{1 + \sin^4(\sqrt{x})}{2\sqrt{x} - 3} \, dx \) diverges by the Direct Comparison Test.

(b) First we have that \( \int_{0}^{\infty} 2x^3 e^{1+x^2} \, dx = \lim_{t \to \infty} \int_{0}^{t} 2x^3 e^{1+x^2} \, dx \) and to evaluate the integral, we use integration by parts with \( u = x^2 \Rightarrow du = 2x \, dx \) and \( dv = 2xe^{1+x^2} \Rightarrow v = e^{1+x^2} \), and so
\[
\int 2x^3 e^{1+x^2} \, dx = x^2 e^{1+x^2} - \int 2xe^{1+x^2} \, dx = x^2 e^{1+x^2} - e^{1+x^2} = e^{1+x^2}(x^2 - 1)
\]
4. (15 pts) The cross sections of a trumpet (perpendicular to the $x$-axis) are circular disks with diameters reaching from the curve $y = -e^x$ to the curve $y = e^x$, $-\infty \leq x \leq \ln 2$ (the trumpet is infinitely long). Find the volume of the trumpet.

**Solution:** Note that the diameter of the cross section is $2e^x$ and so the radius is $r = e^x$, and so the area of the cross section is $A = \pi r^2 = \pi (e^x)^2 = \pi e^{2x}$ and so the volume is

$$V = \int_{-\infty}^{\ln 2} \pi e^{2x} \, dx = \lim_{t \to -\infty} \int_t^{\ln 2} \pi e^{2x} \, dx = \lim_{t \to -\infty} \pi e^{2x} |_t^{\ln 2} = \lim_{t \to -\infty} \pi \frac{e^{2\ln 2} - e^t}{2} = \pi \frac{e^{2\ln 2}}{2} = 2\pi$$

and so the integral **diverges**.

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3. (20 pts, 10 pts ea.) Evaluate the integrals: 

(a) \[ \int_0^{\sqrt{3}} \frac{x^3}{\sqrt{4-x^2}} \, dx \]

(b) \[ \int_0^1 \frac{x + 1}{\sqrt{x^2 + 2x}} \, dx \]

**Solution:**

(a) Using the trig substitution $x = 2 \sin(\theta)$ we have $dx = 2 \cos(\theta) \, d\theta$ and $\sqrt{4-x^2} = 2 \cos(\theta)$ and so

\[
\begin{align*}
\int_0^{\sqrt{3}} \frac{x^3}{\sqrt{4-x^2}} \, dx &= \int_0^{\pi/3} \frac{2^3 \sin^3(\theta)}{2 \cos(\theta)} \, 2 \cos(\theta) \, d\theta \\
&= 8 \int_0^{\pi/3} \sin^3(\theta) \, d\theta \\
&= 8 \int_0^{\pi/3} \sin^2(\theta) \sin(\theta) \, d\theta \\
&= 8 \left[ \frac{1}{2} \int_0^{\pi/3} (1 - \cos^2(\theta)) \sin(\theta) \, d\theta \right] \\
&= \frac{8}{2} \left[ \int_0^{\pi/3} (1 - \cos^2(\theta)) \, d\theta \right] \\
&= 4 \int_0^{\pi/3} (1 - \cos^2(\theta)) \, d\theta \\
&= 4 \left( \frac{1}{2} \int_0^{\pi/3} (1 - \cos(2\theta)) \, d\theta \right) \\
&= 2 \left[ \frac{1}{2} \int_0^{\pi/3} (1 - \cos(2\theta)) \, d\theta \right] \\
&= \frac{1}{2} \left[ \int_0^{\pi/3} (1 - \cos(2\theta)) \, d\theta \right] \\
&= \frac{1}{2} \left[ \frac{\theta}{2} - \frac{1}{4} \sin(2\theta) \right]_0^{\pi/3} \\
&= \frac{1}{2} \left[ \frac{\pi}{6} - \frac{1}{4} \sin( \frac{2\pi}{3} ) - \frac{1}{4} \sin(0) \right] \\
&= \frac{1}{2} \left[ \frac{\pi}{6} - \frac{1}{4} \cdot \frac{\sqrt{3}}{2} \right] \\
&= \frac{\pi}{12} - \frac{\sqrt{3}}{8}
\end{align*}
\]

(b) First we note that \[ \int_0^1 \frac{x + 1}{\sqrt{x^2 + 2x}} \, dx = \lim_{t \to 0^+} \int_t^1 \frac{x + 1}{\sqrt{x^2 + 2x}} \, dx \]

and to evaluate the integral, we use the $u$-substitution $u = x^2 + 2x \Rightarrow du = (2x + 2) \, dx$ and thus,

\[
\int \frac{x + 1}{\sqrt{x^2 + 2x}} \, dx = \frac{1}{2} \int \frac{du}{\sqrt{u}} = \sqrt{u} = \sqrt{x^2 + 2x}
\]

and so

\[
\int_0^1 \frac{x + 1}{\sqrt{x^2 + 2x}} \, dx = \frac{1}{2} \int_{t=0}^1 \frac{du}{\sqrt{u}} = \frac{1}{2} \sqrt{u} = \sqrt{t^2 + 2t} = \sqrt{3}
\]

thus \[ \int_0^1 \frac{x + 1}{\sqrt{x^2 + 2x}} \, dx = \sqrt{3}. \]
5. (a) (15 pts) The region bounded by the curve \( y = \sqrt{x-1}, \) \( 1 \leq x \leq 5 \) and the \( x \)-axis is rotated about the line \( x = 5. \) Find the volume of the resulting solid using the disk/washer method.

(b) (10 pts) The region between the curves \( x = (y + 2)^2 \) and \( x = y + 2 \) is rotated about the line \( y = a \) where \( a \) is a positive constant. Using the disk/washer method, set up, but do not evaluate, an integral to find the volume of this solid.

Solution:

(a) Note that here \( \Delta V = \pi r^2 \Delta y, \) where \( r = 5 - (y^2 + 1) = 4 - y^2 \) and so

\[
V = \int_0^2 \pi (4 - y^2)^2 \, dy = \pi \int_0^2 (16 - 8y^2 + y^4) \, dy = \pi \left[ 16y - \frac{8y^3}{3} + \frac{y^5}{5} \right]_0^2 = \pi \left[ 32 - \frac{64}{3} + \frac{32}{5} \right] = \frac{256\pi}{15}
\]

(b) Here \( \Delta V = \pi [R^2 - r^2] \Delta x \) where, \( R = a - (x - 2) = a + 2 - x \) and \( r = a - (-2 + \sqrt{x}) = a + 2 - \sqrt{x}, \) and note that \( x = (y + 2)^2 \) and \( x = y + 2 \) implies \( x = x^2 \Rightarrow x = 0, 1 \) and so

\[
V = \int_0^1 \pi [(a + 2 - x)^2 - (a + 2 - \sqrt{x})^2] \, dx
\]