1. (a) Consider the graph of $9x^2 + 25y^2 = 225$.

i. (4 pts.) Sketch the curve. Label $x$ and $y$ intercepts and foci.

ii. (3 pts.) The same curve can be defined by the parametric equations $x = a \sin \theta$ and $y = b \cos \theta$, $0 \leq \theta \leq 2\pi$. Find values for the constants $a$ and $b$.

iii. (5 pts.) Use your answer for part (ii) to set up an expression for the length of the curve. Do not evaluate the expression.

Solution:

i. The equation can be written in standard form as $x^2/25 + y^2/9 = 1$, which corresponds to an ellipse with $a = 5$, $b = 3$, and $c = \sqrt{a^2 - b^2} = 4$. The intercepts are $(\pm 5, 0)$ and $(0, \pm 3)$, and the foci are at $(\pm 4, 0)$.

ii. The parametric equations $x = 5 \sin \theta$ and $y = 3 \cos \theta$ trace the ellipse beginning at $(0, 3)$, moving clockwise. (An alternate solution is $a = -5$, $b = -3$.)

iii. $L = \int_{0}^{2\pi} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} \, d\theta = \int_{0}^{2\pi} \sqrt{(5 \cos \theta)^2 + (-3 \sin \theta)^2} \, d\theta$

(b) Let $r = 1 - \sin \theta$.

i. (6 pts.) Find the slope of the curve at $\theta = \pi$.

ii. (5 pts.) Set up but do not evaluate an expression for the length of the curve.

iii. (7 pts.) Find the area of the region that lies inside the curve and outside $r = 1$.

Solution:

i. $\frac{dy}{dx} \bigg|_{\theta=\pi} = \frac{\frac{d}{d\theta} (r \sin \theta)}{\frac{d}{d\theta} (r \cos \theta)} \bigg|_{\theta=\pi} = \frac{(1 - \sin \theta)(\cos \theta) + \sin \theta(-\cos \theta)}{(1 - \sin \theta)(-\sin \theta) + \cos \theta(-\cos \theta)} \bigg|_{\theta=\pi} = \frac{-1 + 0}{0 - 1} = 1$

ii. $L = \int \sqrt{r^2 + (dr/d\theta)^2} \, d\theta = \int_{0}^{2\pi} \sqrt{(1 - \sin \theta)^2 + (-\cos \theta)^2} \, d\theta$

iii.
\[ A = \int \frac{1}{2} (r_1^2 - r_2^2) \, d\theta = \int_\pi^2 \frac{1}{2} ((1 - \sin \theta)^2 - 1^2) \, d\theta = \int_\pi^{2\pi} \frac{1}{2} (-2 \sin \theta + \sin^2 \theta) \, d\theta \\
= \int_\pi^{2\pi} \left( -\sin \theta + \frac{1}{4}(1 - \cos 2\theta) \right) \, d\theta = \left[ \cos \theta + \frac{\theta}{4} - \frac{1}{8} \sin 2\theta \right]_\pi^{2\pi} \\
= \left( 1 + \frac{\pi}{2} - 0 \right) - \left( -1 + \frac{\pi}{4} - 0 \right) = 2 + \frac{\pi}{4} \]

(c) (8 pts.) Match the graphs shown below to the following equations. No explanation is required.

(i) \( r = 1 + \cos \theta \) (ii) \( r = 2 \cos \frac{\theta}{3} \) (iii) \( r = \frac{1}{2} + \cos(3\theta) \) (iv) \( r = \sqrt{1 + \cos^2(4\theta)} \)

Solution: (i) a (ii) d (iii) b (iv) c

2. (a) Evaluate the following integrals.

i. (7 pts.) \( \int t \cos(2t) \, dt \) ii. (7 pts.) \( \int \tan^3 x \, dx \)

Solution:

i. Use integration by parts. Let \( u = t, dv = \cos(2t) \, dt \Rightarrow du = dt, v = \frac{1}{2} \sin(2t) \).

\[ \int t \cos(2t) \, dt = \frac{1}{2} t \sin(2t) - \int \frac{1}{2} \sin(2t) \, dt = \frac{1}{2} t \sin(2t) + \frac{1}{4} \cos(2t) + C \]

ii.

\[ \int \tan^3 x \, dx = \int (\tan^2 x) \tan x \, dx = \int (\sec^2 x - 1) \tan x \, dx \\
= \int \sec^2 x \tan x \, dx - \int \tan x \, dx \\
\]

In the first integral, let \( u = \sec x, du = \sec x \tan x \, dx \). (Alternatively let \( u = \tan x, du = \sec^2 x \, dx \).)

\[ = \int u \, du - (\ln |\sec x| + C_1) = \frac{u^2}{2} - \ln |\sec x| + C \\
= \frac{1}{2} \sec^2 x - \ln |\sec x| + C \text{ or } \frac{1}{2} \tan^2 x + \ln |\cos x| + C \]

(b) Consider the function \( f(x) = \frac{4}{(x - 6)^3} \).

i. (7 pts.) Evaluate \( \int_{6}^{8} f(x) \, dx \).
ii. (5 pts.) What is the error in approximating $\int f(x) \, dx$ on $[8, 10]$ using the Midpoint Rule $M_4$?

iii. (4 pts.) Let $g(x) = \frac{4}{(x - 6)^3(3x^2 + 5)}$. Write the form of the partial fraction decomposition for $g(x)$ but do not solve for the numerical coefficients.

**Solution:**

i. This is an improper integral.

\[
\int_6^8 \frac{4}{(x - 6)^3} \, dx = \lim_{t \to 6^+} \int_t^8 \frac{4}{(x - 6)^3} \, dx = \lim_{t \to 6^+} \left[-2(x - 6)^{-2}\right]_t^8 = -2 \lim_{t \to 6^+} \left(\frac{1}{2^2} - \frac{1}{(t - 6)^2}\right) = \infty
\]

The integral is **divergent**.

ii. First find an upper bound for $|f''|$: $f(x) = 4(x - 6)^{-3} \Rightarrow f'(x) = -12(x - 6)^{-4} \Rightarrow f''(x) = 48(x - 6)^{-5}$. Since $f''$ is a decreasing function, the maximum value of $|f''|$ is $K = f''(8) = 48(2)^{-5}$.

\[
|E_M| \leq K(b - a)^3 \frac{2^n}{24n^2} = \frac{48}{2^5} \cdot \frac{(10 - 8)^3}{24(4^2)} = \frac{1}{32}
\]

iii. $g(x) = \frac{A}{x - 6} + \frac{B}{(x - 6)^2} + \frac{C}{(x - 6)^3} + \frac{Dx + E}{3x^2 + 5}$.

3. (a) Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{(-1)^n}{2^n3^n}$.

i. (6 pts.) Is $\{a_n\}$ monotonic? Is $\{a_n\}$ bounded?

ii. (6 pts.) Let $S$ represent the sum of the series and let $s_n$ represent the $n$th partial sum. Estimate the value of $|S - s_2|$ without finding $S$.

iii. (6 pts.) Now find the sum $S$ of the series.

**Solution:**

i. $a_n$ is an alternating sequence and therefore **not monotonic**. Since $|a_n| = 1/(2^n3^n)$ is decreasing as $n \to \infty$, the sequence $\{a_n\}$ is bounded by $a_1 = -1/6$ and above by $a_2 = 1/36$, and therefore **bounded**.

ii. By the Alternating Series Estimation Theorem, since $|a_n|$ is decreasing and $\lim_{n \to \infty} |a_n| = 0$, $|S - s_2| \leq |a_3| = \frac{1}{216}$.

iii. This is a geometric series with ratio $r = -1/6$ and first term $a = -1/6$ so the sum of the series is $S = \frac{a}{1 - r} = \frac{-1/6}{1 + 1/6} = \frac{1}{7}$.

(b) Are the following series absolutely convergent, conditionally convergent, or divergent?

i. (7 pts.) $\sum_{n=4}^{\infty} \frac{n}{\sqrt{n^3 - 1}}$

ii. (7 pts.) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{100^n}{n^n}$

**Solution:**
i. By the Direct Comparison Test, since \( \frac{n}{\sqrt{n^3 - 1}} > \frac{n}{\sqrt{n^3}} = \frac{1}{\sqrt{n}} \) and \( \sum \frac{1}{\sqrt{n}} \) is a divergent p-series with \( p = 1/2 < 1 \), the given series is \( \text{divergent} \).

Alternative Solution: Compare to the divergent p-series \( \sum \frac{1}{\sqrt{n}} \). By the Limit Comparison Test, since \( \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n}{\sqrt{n^3 - 1}} \cdot \frac{\sqrt{n^3}}{1} = \lim_{n \to \infty} \sqrt{\frac{n^3}{n^3 - 1}} = 1 \), the given series is also \( \text{divergent} \).

ii. By the Root Test, since \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{100}{n} = 0 \), the series is \( \text{absolutely convergent} \).

Alternative Solution: Use the Ratio Test.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{100^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{100^n} \right| = \lim_{n \to \infty} \frac{100}{n+1} \left( \frac{n}{n+1} \right)^n = \lim_{n \to \infty} \frac{100}{e(n+1)} = 0
\]

since \( \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n = \frac{1}{e} \). Therefore the series is \( \text{absolutely convergent} \).

4. (a) Suppose \( g(x) \) equals the power series \( \sum_{n=2}^{\infty} \frac{(n+1)(x+b)^n}{c^{2n}} \), where \( b \) and \( c \) are constants, and the series has an interval of convergence of \( -6 < x < 2 \).

i. (3 pts.) Find the center and radius of convergence of the series.

ii. (4 pts.) Evaluate \( \int g(x) \, dx \) as a power series.

iii. (7 pts.) Given the interval of convergence, find possible values for \( b \) and \( c \). Justify your answer using appropriate test(s).

Solution:

i. The interval \( -6 < x < 2 \) corresponds to the interval \( |x-a| < R \) with center \( a = -2 \) and radius \( R = 4 \).

ii. \( \int g(x) \, dx = \int \sum_{n=2}^{\infty} \frac{(n+1)(x+b)^n}{c^{2n}} \, dx = C + \sum_{n=2}^{\infty} \frac{(n+1)(x+b)^{n+1}}{c^{2n}n+1} = C + \sum_{n=2}^{\infty} \frac{(x+b)^{n+1}}{c^{2n}} \)

iii. A power series \( \sum c_n(x-a)^n \) converges for \( |x-a| < R \). Since the center of the given series is \( a = -2 \), the constant \( b = 2 \). Apply the Ratio Test to find the constant \( c \).

\[
L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+2)(x+2)^{n+1}}{c^{2n+2}} \cdot \frac{c^{2n}}{(n+1)(x+2)^n} \right| = \lim_{n \to \infty} \frac{n+2}{n+1} \cdot \frac{x+2}{c^2} = \left| \frac{x+2}{c^2} \right| < 1 \Rightarrow |x+2| < c^2.
\]

This interval corresponds to \( |x-a| < R \) so \( c^2 = R = 4 \) \( \Rightarrow c = 2 \) or \( -2 \) for absolute convergence.

At the endpoints of the interval, \( x = -6 \) and \( x = 2 \), the power series is divergent because both series \( \sum (-n+1) \) and \( \sum (n+1) \) diverge by the Test for Divergence.

(b) (5 pts.) Use series to evaluate \( \lim_{x \to 0} \frac{e^{2x} - 1 - 2x - 2x^2}{8x^3} \).
1. Consider the region bounded by
   \( m \) and \( M \).
   \( (b) \) (6 pts.) Masses
   \( (c) \) (6 pts.) Solve the initial value problem
   \( (c) \) (7 pts.) Find
   \( \int \)
   \( f \) is \( f^{(n)}(x) = \frac{(-1)^n n!}{(x+3)^{n+1}} \) for \( n = 1, 2, \ldots, x \neq -3. \)
   \( T_2(x) = \frac{1}{4} + \frac{f'(1)}{1!} (x-1) + \frac{f''(1)}{2!} (x-1)^2 = \frac{1}{4} - \frac{1}{16} (x-1) + \frac{1}{64} (x-1)^2 \)

5. (a) Consider the region bounded by \( x = 1 - y^2 \) and \( x = 0. \)
   \( i. \) (6 pts.) Set up but do not evaluate an expression for the volume of the solid obtained by rotating the region about the line \( y = -2. \)
   \( ii. \) (6 pts.) Set up but do not evaluate an expression for the surface area obtained by rotating \( x = 1 - y^2 \) about the \( x \)-axis, \( 0 \leq y \leq 1. \)
   \( \text{Solution: Note that the curve can be expressed as } y = \pm \sqrt{1-x}, 0 \leq x \leq 1. \)
   \( \text{i. Shell Method: } V = \int_{-1}^{1} 2\pi rh \ dy = \int_{-1}^{1} 2\pi (y+2) (1-y^2) \ dy \)
   \( \text{Washer Method: } V = \int_{0}^{1} \pi (R^2 - r^2) \ dx = \int_{0}^{1} \pi ((\sqrt{1-x}+2)^2 - (\sqrt{1-x}+2)^2) \ dx \)
   \( \text{ii. } S = \int_{0}^{1} 2\pi r \ ds \text{ where } ds = \sqrt{1+(dy/dx)^2} \ dx \text{ or } ds = \sqrt{1+(dx/dy)^2} \ dy \)
   \( S = \int_{0}^{1} 2\pi y \sqrt{1+4y^2} \ dy \text{ or } \int_{0}^{1} 2\pi \sqrt{1-x} \sqrt{1+\frac{1}{4(1-x)}} \ dx \)

(b) (6 pts.) Masses \( m_1 = 4, m_2 = 1, m_3 = 5 \) are located at the points \( P_1(3, 1), P_2(-2, 0), P_3(0, -4) \), respectively. Find the moments \( M_x \) and \( M_y \) and the center of mass of the system.
   \( \text{Solution: } M_x = \sum m_i y_i = 4(1) + 1(0) + 5(-4) = -16 \)
   \( M_y = \sum m_i x_i = 4(3) + 1(-2) + 5(0) = 10 \)
   \( m = \sum m_i = 4 + 1 + 5 = 10 \)
   \( (\bar{x}, \bar{y}) = (M_y/m, M_x/m) = (10/10, -16/10) = (1, -8/5) \)

(c) (6 pts.) Solve the initial value problem \( y' = 2x \sqrt{1-y^2}, \ y(0) = \frac{1}{2}. \)
   \( \text{Solution: } \)
   \( \frac{dy}{dx} = 2x \sqrt{1-y^2} \)
   \( \int \frac{dy}{\sqrt{1-y^2}} = \int 2x \ dx \)
   \( \sin^{-1} y = x^2 + C \text{ for } |x^2 + C| \leq \frac{\pi}{2} \)
Now use the initial value to find $C$.

\[
\sin^{-1} \frac{1}{2} = 0 + C \Rightarrow C = \frac{\pi}{6}
\]

\[
\sin^{-1} y = x^2 + \frac{\pi}{6}
\]

\[
y = \sin \left( x^2 + \frac{\pi}{6} \right) \quad \text{for} \quad |x| \leq \sqrt{\frac{\pi}{3}}.
\]