1. (12 points) Consider the area bounded by \( y = xe^{-x}, y = 0, \) and \( x = 2 \). Set up, but do not evaluate, the integrals to calculate the following:

(a) The volume generated by rotating this region around the \( y \)-axis.

(b) The volume generated by rotating this region around the line \( y = 4 \).

Solution:

(a) The cylindrical shells method yields

\[
V = \int 2\pi rh \, dx = \int_0^2 2\pi x (xe^{-x}) \, dx.
\]

(b) The washer method yields

\[
V = \int \pi (R^2 - r^2) \, dx = \int_0^2 \pi \left[ 4^2 - (4 - xe^{-x})^2 \right] \, dx.
\]

2. The following parts are not related:

(a) (10 points) Let \( y = \cosh(x) \) for \( 0 \leq x \leq 2 \). Find the surface area of rotation of the curve around the \( y \)-axis.
Solution: Use the identity \( \cosh^2 x - \sinh^2 x = 1 \Rightarrow \cosh x = \sqrt{1 + \sinh^2 x} \) and apply integration by parts.

\[
ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \sqrt{1 + \sinh^2 x} \, dx = \cosh x \, dx
\]

\[
S = \int 2\pi r \, ds = \int_0^2 2\pi x \cosh x \, dx = 2\pi \int_0^2 \frac{x}{\sqrt{1 + \sinh^2 x}} \cosh x \, dx\,
\]

\[
= 2\pi \left( x \sinh x \bigg|_0^2 - \int_0^2 \sinh x \, dx \right) = 2\pi \left( 2 \sinh 2 - 0 - \left[ \cosh x \right]_0^2 \right)
\]

\[
= 2\pi \left( 2 \sinh 2 - (\cosh 2 - \cosh 0) \right) = 2\pi \left( 2 \sinh 2 - \cosh 2 + 1 \right)
\]

\[
= \pi (e^2 - 3e^{-2} + 2)
\]

(b) (12 points) Find the center of mass of a thin plate with uniform density \( \rho \) that covers the region bounded by \( y = \sin x \) and the \( x \)-axis for \( 0 \leq x \leq \pi \).

Solution: By symmetry the \( x \)-coordinate of the center of mass is \( \overline{x} = \frac{\pi}{2} \).

The \( y \)-coordinate \( \overline{y} = M_x/m \). First find the mass \( m \).

\[
m = \rho A = \rho \int_0^\pi \sin x \, dx = \rho \left[ -\cos x \right]_0^\pi = \rho (1 + 1) = 2\rho
\]

Next find the moment about the \( x \)-axis. Use the identity \( \sin^2 x = (1 - \cos 2x)/2 \).

\[
M_x = \rho \int_0^\pi \frac{1}{2} [f(x)]^2 \, dx = \rho \int_0^\pi \frac{1}{2} \sin^2 x \, dx = \rho \int_0^\pi \frac{1}{4} (1 - \cos 2x) \, dx
\]

\[
= \frac{\rho}{4} \left[ x - \frac{1}{2} \sin 2x \right]_0^\pi = \frac{\rho}{4} (\pi - 0) = \frac{\pi}{4} \rho
\]

The \( y \)-coordinate is

\[
\overline{y} = \frac{M_x}{m} = \frac{\frac{\pi}{4} \rho}{2 \rho} = \frac{\pi}{8}.
\]

The center of mass \((\overline{x}, \overline{y}) = \left( \frac{\pi}{2}, \frac{\pi}{8} \right)\).

3. (14 points) Suppose \( dy/dx = xy \) with \( y \left( \frac{1}{2} \right) = e \).

(a) Solve the differential equation with the given initial condition for the function \( y \).

(b) Which of the following direction fields matches the differential equation?
Solution:

(a) Separate the variables and integrate.

\[
\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x \, dx \Rightarrow \int \frac{dy}{y} = \int x \, dx \Rightarrow \ln |y| = \frac{x^2}{2} + C
\]

Use the initial value to solve for \( C \).

\[
\ln e = \frac{1}{2} \cdot \frac{1}{4} + C \Rightarrow C = 1 - \frac{1}{8} = \frac{7}{8}
\]

Now solve for \( y \).

\[
\ln |y| = \frac{x^2}{2} + \frac{7}{8} \Rightarrow y = e^\left(\frac{x^2}{2} + \frac{7}{8}\right)
\]

(b) Direction field (1) matches the given differential equation. Notice that the slopes equal 0 whenever \( x = 0 \) or \( y = 0 \).

4. (16 points) Consider the sequence given by \( a_1 = \pi \) and \( a_i = \frac{a_{i-1}}{13} - \frac{\pi}{13} \) for \( i = 1, 2, 3, \ldots \) and the series \( \sum_{i=1}^{\infty} a_i \).

(a) Find a direct expression (not recursive) for \( a_2, a_3, \) and \( a_i \).

(b) Does the sequence \( \{a_i\} \) converge? If so, what is its limit?

(c) What is \( s_3 \), the third partial sum of the series \( \sum_{i=1}^{\infty} a_i \)? (You need not simplify your answer.)

(d) Find a simple expression for \( s_n \), the \( n^{th} \) partial sum of the series. Does the sequence \( \{s_n\} \) converge? If so, what is its limit?

(e) Does the series \( \sum_{i=1}^{\infty} a_i \) converge? If so, what is its limit?

Solution:
(a) \(a_1 = \pi, a_2 = a_1 \frac{\pi}{13} = \frac{\pi^2}{13}, a_3 = a_2 \frac{\pi}{13} = \frac{\pi^3}{13^2}, a_i = \frac{\pi^i}{13^{i-1}}\)

(b) The sequence \(\{r^i\}\) converges to 0 for \(|r| < 1\). Because \(\{a_i\} = \left\{\frac{\pi^i}{13^{i-1}}\right\} = \left\{\pi \left(\frac{\pi}{13}\right)^{i-1}\right\}\) has a ratio of \(r = \frac{\pi}{13} < 1\), the sequence converges to 0.

(c) The third partial sum is
\[s_3 = a_1 + a_2 + a_3 = \pi + \frac{\pi^2}{13} + \frac{\pi^3}{13^2}.\]

(d) Either use the formula \(s_n = a(1 - r^n)/(1 - r)\) or solve for \(s_n\) as follows:
\[
s_n = \pi + \frac{\pi^2}{13} + \frac{\pi^3}{13^2} + \cdots + \frac{\pi^n}{13^{n-1}}
\]
\[
\frac{\pi}{13} s_n = \frac{\pi^2}{13} + \frac{\pi^3}{13^2} + \frac{\pi^4}{13^3} + \cdots + \frac{\pi^{n+1}}{13^n}
\]
Subtract the two equations.
\[
s_n - \frac{\pi}{13} s_n = \pi - \frac{\pi^{n+1}}{13^n}
\]
\[
s_n \left(1 - \frac{\pi}{13}\right) = \pi \left(1 - \left(\frac{\pi}{13}\right)^n\right)
\]
\[
s_n = \frac{\pi \left(1 - \left(\frac{\pi}{13}\right)^n\right)}{1 - \frac{\pi}{13}}
\]

The infinite series \(\sum a_i\) is a geometric series with \(r = \frac{\pi}{13} < 1\) so \(\{s_n\}\) converges. Because \(\{(\frac{\pi}{13})^n\}\) converges to 0, the sequence \(\{s_n\}\) converges to
\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{\pi \left(1 - \left(\frac{\pi}{13}\right)^n\right)}{1 - \frac{\pi}{13}} = \frac{\pi(1)}{1 - \frac{\pi}{13}} = \frac{13\pi}{13 - \pi}
\]

5. (12 points) Determine whether the following sequences are convergent or divergent. If they are convergent, find the limit. Explain your work and name any test or theorem you use.

(a) \(b_n = \frac{\sin \left(\frac{n\pi}{6}\right)}{6^n}\)

(b) \(c_n = \frac{(n + 1)!}{(n - 1)!}\)

Solution:

(a) Use the Squeeze Theorem. Because \(-1 \leq \sin \theta \leq 1\) for all \(\theta\),
\[
-1 \leq \sin \left(\frac{n\pi}{6}\right) \leq 1
\]
\[
-\frac{1}{6^n} \leq \frac{\sin \left(\frac{n\pi}{6}\right)}{6^n} \leq \frac{1}{6^n}
\]
The sequence \(\{r^n\}\) converges to 0 for \(|r| < 1\) so \(\{(\frac{1}{6})^n\}\) converges to 0. By the Squeeze Theorem the sequence \(\{b_n\} = \left\{\frac{\sin(n\pi/6)}{6^n}\right\}\) also converges to 0.
6. (24 points) Determine whether the following infinite series are convergent or divergent. Explain your work and name any test or theorem you use.

(a) \[ \frac{1}{2\sqrt{2} + 1} + \frac{1}{3\sqrt{3} + 1} + \frac{1}{4\sqrt{4} + 1} + \frac{1}{5\sqrt{5} + 1} + \cdots \]

(b) \[ \sum_{n=1}^{\infty} \ln \left(1 + \frac{2}{n}\right) \]

(c) \[ \sum_{k=2}^{\infty} \frac{1}{k \ln k} \]

Solution:

(a) \[ \frac{1}{2\sqrt{2} + 1} + \frac{1}{3\sqrt{3} + 1} + \frac{1}{4\sqrt{4} + 1} + \frac{1}{5\sqrt{5} + 1} + \cdots = \sum_{n=2}^{\infty} \frac{1}{n \sqrt{n} + 1} = \sum_{n=2}^{\infty} \frac{1}{n^{3/2} + 1} \]

Use the Comparison Test. The p-series \( \sum_{n=2}^{\infty} \frac{1}{n^{3/2}} \) is convergent with \( p = \frac{3}{2} \). Because \( \frac{1}{n^{3/2} + 1} < \frac{1}{n^{3/2}} \), the series \( \sum_{n=2}^{\infty} \frac{1}{n^{3/2} + 1} \) also is convergent.

(b) First apply the Test for Divergence: \( \lim_{n \to \infty} \ln \left(1 + \frac{2}{n}\right) = \ln 1 = 0 \).

Note that this is a telescoping series. The terms can be written as
\[ \sum_{n=1}^{\infty} \ln \left(1 + \frac{2}{n}\right) = \sum_{n=1}^{\infty} \ln \left(\frac{n+2}{n}\right) = \sum_{n=1}^{\infty} \left(\ln(n+2) - \ln(n)\right) \]

The \( n \)th partial sum is
\[ s_n = (\ln 3 - \ln 1) + (\ln 4 - \ln 2) + (\ln 5 - \ln 3) + \cdots + (\ln(n+1) - \ln(n+2)) + (\ln(n+2) - \ln n) \]
\[ = -\ln 1 - \ln 2 + \ln(n+1) + \ln(n+2) \]
\[ = -\ln 2 + \ln(n+1) + \ln(n+2). \]

Since \( \lim_{n \to \infty} s_n = \lim_{n \to \infty} (-\ln 2 + \ln(n+1) + \ln(n+2)) = \infty \), the series is divergent.

(c) Use the Integral Test. The function \( f(x) = 1/(x \ln x) \) is positive and continuous. The derivative \( f'(x) = -(x \ln x)^{-2}(1 + \ln x) < 0 \) so \( f \) is decreasing.

\[ \int_{2}^{\infty} \frac{dx}{x \ln x} = \lim_{t \to \infty} \int_{2}^{t} \frac{dx}{x \ln x} = \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{du}{u} = \lim_{t \to \infty} \ln |u| \bigg|_{\ln 2}^{\ln t} = \lim_{t \to \infty} (\ln(\ln t) - \ln(\ln 2)) = \infty \]
Because the integral is divergent, the series \( \sum_{k=2}^{\infty} \frac{1}{k \ln k} \) also is \[\text{divergent}\].

**Formulas**

\[
\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad \cosh^2 x - \sinh^2 x = 1
\]