1. (22 points) The velocity function (in meters/second) for a particle moving along a line is \( v(t) = 8 - t^3 \). For the interval \( 0 \leq t \leq 4 \), evaluate the following:

(a) the general indefinite integral of \( v(t) \),
(b) the definite integral of \( v(t) \),
(c) the definite integral of \( |v(t)| \),
(d) the average value of \( v(t) \),
(e) the value of \( c \) that satisfies the Mean Value Theorem for Integrals,
(f) the displacement of the particle,
(g) the distance traveled by the particle.

Solution:

(a) \[ \int v(t) \, dt = \int (8 - t^3) \, dt = 8t - \frac{t^4}{4} + C \]

(b) \[ \int_0^4 v(t) \, dt = \int_0^4 (8 - t^3) \, dt = 8t - \frac{t^4}{4} \bigg|_0^4 = 8(4) - 64 - 0 = 32 - 64 = -32 \]

(c) First determine that \( v(t) = 8 - t^3 = 0 \) at \( t = 2 \).

\[ \int_0^4 |v(t)| \, dt = \left| \int_0^2 (8 - t^3) \, dt \right| + \left| \int_2^4 (8 - t^3) \, dt \right| \\
= \left| 8t - \frac{t^4}{4} \bigg|_0^2 \right| + \left| 8t - \frac{t^4}{4} \bigg|_2^4 \right| \\
= \left| 8(2) - 4 - 0 \right| + \left| 8(4) - 64 - (8(2) - 4) \right| \\
= 12 + |32 - 12| = 12 + 44 = 56 \]

(d) From part (b) we know that \( \int_0^4 v(t) \, dt = -32 \). Then

\[ v_{\text{ave}} = \frac{1}{4 - 0} \int_0^4 v(t) \, dt = \frac{1}{4}(-32) = -8 \]

(e) The Mean Value Theorem states that there is a \( c \) in \([0, 4]\) such that

\[ v(c) = v_{\text{ave}} \]

\[ 8 - c^3 = -8 \]

\[ c^3 = 16 \]

\[ c = \sqrt[3]{16} = 2\sqrt[3]{2} \]

(f) The displacement equals \( \int_0^4 v(t) \, dt = -32 \) (from part (b)).

(g) The distance traveled equals \( \int_0^4 |v(t)| \, dt = 56 \) (from part (c)).

2. (36 points) Evaluate the following integrals.

(a) \[ \int (\sin x) \sec^2 (\cos x) \, dx \]

(b) \[ \int_0^9 \left( \sqrt{81 - x^2} - 4x \right) \, dx \]

(c) \[ \int 3t^3 \sqrt{t^2 - 3} \, dt \]

(d) \[ \int_0^{\pi/4} \theta \sec^2 (\theta^2) \tan (\theta^2) \, d\theta \]

Solution:

(a) (Substitution Rule) Let \( u = \cos x \), so \( du = -\sin x \, dx \). Thus,

\[ \int (\sin x) \sec^2 (\cos x) \, dx = - \int \sec^2 (\cos x) (-\sin x) \, dx \]

\[ = - \int \sec^2 u \, du \]
\[
\int_0^9 \left( \sqrt{81 - x^2} - 4 \sqrt{x} \right) \, dx = \int_0^9 \sqrt{81 - x^2} \, dx - \int_0^9 4x^{1/2} \, dx
\]
\[
= \frac{1}{4} \left( \pi \left( 9^2 \right) \right) - \left[ 4 \cdot \frac{2}{3} x^{3/2} \right]_0^9
\]
\[
= \frac{81\pi}{4} - \frac{8}{3} \left( 9^{3/2} - 0 \right)
\]
\[
= \frac{81\pi}{4} - \frac{8}{3} (27) = \frac{81\pi}{4} - 72
\]

(c) (Substitution Rule) Let \( u = t^2 - 3 \), so \( du = 2t \, dt \) and \( t^2 = u + 3 \). Thus,
\[
\int 3t^3 \sqrt{t^2 - 3} \, dt = \frac{3}{2} \int t^2 \sqrt{t^2 - 3} - 3 (2t) \, dt
\]
\[
= \frac{3}{2} \int (u + 3)u^{1/3} \, du
\]
\[
= \frac{3}{2} \left[ u^{4/3} + 3u^{1/3} \right] \, du
\]
\[
= \frac{3}{2} \left[ \frac{3}{7} u^{7/3} + 3 \cdot \frac{3}{4} u^{1/3} \right] + C
\]
\[
= \frac{9}{14} u^{7/3} + \frac{27}{8} u^{1/3} + C
\]
\[
= \frac{9}{14} (t^2 - 3)^{7/3} + \frac{27}{8} (t^2 - 3)^{1/3} + C
\]

(d) (Substitution Rule) Let \( u = \tan(\theta^2) \), so \( du = 2\theta \sec^2(\theta^2) \, d\theta \) (chain rule). The limits are then \( u(\sqrt{\pi/4}) = \tan(\pi/4) = 1 \), \( u(0) = \tan(0) = 0 \). Thus,
\[
\int_{\sqrt{\pi/4}}^{\pi/4} \theta \sec^2(\theta^2) \tan(\theta^2) \, d\theta = \frac{1}{2} \int_{\sqrt{\pi/4}}^{\pi/4} \tan(\theta^2)(2\theta) \sec^2(\theta^2) \, d\theta
\]

Alternate solution: Let \( u = \sec(\theta^2) \), so \( du = 2\theta \sec(\theta^2) \tan(\theta^2) \, d\theta \). The limits are then \( u(\sqrt{\pi/4}) = \sec(\pi/4) = \sqrt{2} \), \( u(0) = \sec(0) = 1 \). Thus,
\[
\int_{\sqrt{\pi/4}}^{\pi/4} \theta \sec^2(\theta^2) \tan(\theta^2) \, d\theta = \frac{1}{2} \int_{\sqrt{\pi/4}}^{\pi/4} \sec(\theta^2)(2\theta) \sec(\theta^2) \tan(\theta^2) \, d\theta
\]
\[
= \frac{1}{2} \int_{\sqrt{2}}^{\sqrt{\pi/4}} u \, du = \frac{1}{2} \left[ \frac{1}{2} u^2 \right]_{\sqrt{2}}^{\sqrt{\pi/4}}
\]
\[
= \frac{1}{4} \int_{2}^{1} |2 - 1| = \frac{1}{4}
\]

3. (15 points) Let \( g(x) = \frac{2}{x^3 - 6} \) for \( x > \sqrt{6} \).

(a) Show that \( g \) is one-to-one and thus invertible.
(b) Use \( g' \) to find \( (g^{-1})' \) (1). (Find the value without finding \( g^{-1} \)).
(c) Find \( g^{-1}(x) \).
(d) What are the domain and range of \( g^{-1} \)?

Solution:

(a) \( g'(x) = \frac{-2(3x^2)}{(x^3 - 6)^2} = \frac{-6x^2}{(x^3 - 6)^2} < 0 \) for all values of \( x \) for which \( g(x) \) is defined. Since it is always decreasing the function is one-to-one.

(b) First, we find \( g^{-1}(1) \):
\[
x^3 - 6 = 2
\]
\[
x = \sqrt[3]{8} = 2
\]

Alternate solution: Let \( u = \sec(\theta^2) \), so \( du = 2\theta \sec(\theta^2) \tan(\theta^2) \, d\theta \). The limits are then \( u(\sqrt{\pi/4}) = \sec(\pi/4) = \sqrt{2} \), \( u(0) = \sec(0) = 1 \). Thus,
\[
\int_{\sqrt{\pi/4}}^{\pi/4} \theta \sec^2(\theta^2) \tan(\theta^2) \, d\theta = \frac{1}{2} \int_{\sqrt{\pi/4}}^{\pi/4} \sec(\theta^2)(2\theta) \sec(\theta^2) \tan(\theta^2) \, d\theta
\]
\[
= \frac{1}{2} \int_{\sqrt{2}}^{\sqrt{\pi/4}} u \, du = \frac{1}{2} \left[ \frac{1}{2} u^2 \right]_{\sqrt{2}}^{\sqrt{\pi/4}}
\]
\[
= \frac{1}{4} \int_{2}^{1} |2 - 1| = \frac{1}{4}
\]
\[ x^3 = 8 \]
\[ x = 2. \]

So \( g^{-1}(1) = 2. \)

\[
(g^{-1})'(1) = \frac{1}{g'(g^{-1}(1))} = \frac{1}{g'(2)}
= \frac{1}{-6(2^2)} = \frac{1}{-6(4)}
= \frac{4}{-24} = \frac{-1}{6}.
\]

(c) To find \( g^{-1}(x) \):

\[
y = \frac{2}{x^3 - 6}
\]
\[
g(x^3 - 6) = 2
\]
\[
(x^3)y = 2 + 6y
\]
\[
x = \frac{3}{2}y + 6
\]
\[
g^{-1}(x) = \sqrt[3]{\frac{2}{x} + 6}.
\]

(d) The domain of \( g^{-1} \) is the range of \( g \): \((0, \infty)\). The range of \( g^{-1} \) is the domain of \( g \): \((\sqrt{6}, \infty)\).

4. (15 points) Consider the function \( f(t) = \frac{\cos t}{t} \), shown below, and the function

\[
g(x) = \int_{\frac{\pi}{2}}^{x} f(t) \, dt, \quad \frac{\pi}{2} \leq x \leq 4\pi.
\]

(a) Find \( g'(x) \) and \( g'(2\pi) \).
(b) On which interval(s) is \( g \) decreasing?
(c) At what value(s) of \( x \) do the local minimum values of \( g \) occur?
(d) Where does \( g \) attain its absolute minimum value?
(e) Does \( g \) have an inflection point at \( x = \frac{3\pi}{2} \)? Explain.

\[
f(t) = \frac{\cos t}{t}
\]

\[
\begin{align*}
4 & \quad \pi & \quad \frac{5\pi}{2} & \quad \frac{7\pi}{2} & \quad 4\pi \\
\end{align*}
\]

Solution:

(a) \( g'(x) = \frac{\cos x}{x} \) by FTC-I.
\[
g'(2\pi) = \frac{\cos 2\pi}{2\pi} = \frac{1}{2\pi}
\]

(b) \( g \) is decreasing when \( f < 0 \) on \( \left( \frac{\pi}{2}, \frac{3\pi}{2} \right), \left( \frac{5\pi}{2}, \frac{7\pi}{2} \right) \).

(c) \( g \) has local minimum values when \( g \) changes from decreasing to increasing at \( x = \frac{3\pi}{2}, \frac{7\pi}{2} \).

(d) We check the critical numbers of \( g \) at \( x = \frac{3\pi}{2}, \frac{7\pi}{2} \) and the endpoints of the interval at \( x = \frac{\pi}{2}, 4\pi \). The absolute minimum value of \( g \) occurs at \( x = \frac{3\pi}{2} \).
Since \( g'' = f' \), we examine the slope of \( f \) and find that \( g'' \left( \frac{3\pi}{2} \right) = f' \left( \frac{3\pi}{2} \right) > 0 \) so \( g \) is concave up there and has no inflection point at \( x = \frac{3\pi}{2} \).

5. (12 points)

(a) If \( g \) is continuous on \([-3, 3]\) and \( \int_{-3}^{-1} g(x) \, dx = -2 \) and \( \int_{0}^{3} g(x) \, dx = 4 \), find the value of \( \int_{-1}^{0} g(x) \, dx \) for the following cases:
   i. \( g \) is even,
   ii. \( g \) is odd,
   iii. the average value of \( g \) on \([-3, 3]\) is \(-1\).

(b) A comparison property of the integral states that if \( m \leq f(x) \leq M \) for \( a \leq x \leq b \), then

\[
    m(b - a) \leq \int_{a}^{b} f(x) \, dx \leq M(b - a).
\]

Use this property to find the values of \( L \) and \( U \) if

\[
    L \leq \int_{-\pi}^{\pi} \left( \sin^4 x + 2 \sin^2 x \right) \, dx \leq U.
\]

Solution:

(a) i. If \( g \) is even, then \( \int_{-a}^{0} g(x) \, dx = \int_{0}^{a} g(x) \, dx. \)

\[
    \int_{-3}^{-1} g(x) \, dx = \int_{0}^{3} g(x) \, dx = 4
    \int_{-3}^{-1} g(x) \, dx + \int_{0}^{3} g(x) \, dx = 4
\]

(b) We find the minimum and maximum values of \( f(x) = \sin^4 x + 2 \sin^2 x \) on \([-\pi, \pi]\). Since \(-1 \leq \sin x \leq 1\), then

\[
    0 \leq \sin^2 x \leq 1,
\]
$0 \leq \sin^4 x \leq 1$.

It follows that the minimum value of $f$ is $m = 0$ and the maximum value of $f$ is $M = 1 + 2(1) = 3$. Then

\begin{align*}
L &= m(b - a) = 0 \\
U &= M(b - a) = 3(\pi - (-\pi)) = 6\pi.
\end{align*}