1. (28 points) Evaluate the following limits.

(a) \( \lim_{x \to 5^-} \frac{x^2 - 25}{x^2 - 10x + 25} \)

(b) \( \lim_{t \to 3^+} \frac{3 - t}{2\sqrt{3} - \sqrt{4t}} \)

(c) \( \lim_{\theta \to 0} 5\theta^2 \cot \theta \csc(4\theta) \)

(d) \( \lim_{x \to \infty} \frac{2x - 8x^3}{4 + 3x^2} \)

Solution:

(a) \( \lim_{x \to 5^-} \frac{x^2 - 25}{x^2 - 10x + 25} = \lim_{x \to 5^-} \frac{(x + 5)(x - 5)}{(x - 5)^2} = \lim_{x \to 5^-} \frac{x + 5}{x - 5} = \frac{10}{0^-} = -\infty \)

(b) \( \lim_{t \to 3^+} \frac{3 - t}{2\sqrt{3} - \sqrt{4t}} \cdot \frac{2\sqrt{3} + \sqrt{4t}}{2\sqrt{3} + \sqrt{4t}} = \lim_{t \to 3^+} \frac{(3 - t)(2\sqrt{3} + \sqrt{4t})}{12 - 4t} = \lim_{t \to 3^+} \frac{(3 - t)(2\sqrt{3} + \sqrt{4t})}{4(3 - t)} = \frac{2\sqrt{3} + \sqrt{4}}{4} = \frac{2\sqrt{3} + 2}{4} = \sqrt{3} \)

(c) Use the theorem \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \lim_{\theta \to 0} \frac{\theta}{\sin \theta} = 1. \)

\( \lim_{\theta \to 0} 5\theta^2 \cot \theta \csc(4\theta) = \lim_{\theta \to 0} 5\theta^2 \cdot \frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\sin 4\theta} = \lim_{\theta \to 0} 5\theta^2 \cdot \frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\sin \theta} \cdot \frac{\theta}{\sin \theta} \cdot \frac{4\theta}{4\theta} = \lim_{\theta \to 0} 5\theta^2 \cdot 1 \cdot 1 \cdot \frac{1}{\theta} \cdot \frac{1}{4\theta} = \frac{5}{4} \)

(d) \( \lim_{x \to \infty} \frac{2x - 8x^3}{4 + 3x^2} = \lim_{x \to \infty} \frac{2x}{4x^2 + 3} = \lim_{x \to \infty} \frac{-8x}{3} = -\infty \)

2. (18 points)

\( g(x) = \begin{cases} 
  x^3 - x + a, & x \leq \pi \\
  3 + \cos x, & x > \pi 
\end{cases} \)

(a) For what value of \( a \) is \( g \) continuous at \( x = \pi \)? Use the definition of continuity to justify your answer.

(b) Given your answer for (a), show that \( g(x) = 0 \) has at least one solution. Indicate an interval where the solution can be found.

Solution:

(a) \( g \) is continuous at \( x = \pi \) if \( g(\pi) = \lim_{x \to \pi^-} g(x) = \lim_{x \to \pi^+} g(x) \).

\( g(\pi) = \lim_{x \to \pi^-} (x^3 - x + a) = \pi^3 - \pi + a \) and \( \lim_{x \to \pi^+} (3 + \cos x) = 3 - 1 = 2. \) Set the two values equal: \( \pi^3 - \pi + a = 2 \Rightarrow a = 2 - \pi^3 + \pi \).
Both $x^3 - x + a$ and $3 + \cos x$ are continuous functions. If $a = 2 - \pi^3 + \pi$, then $g$ is continuous everywhere. Since $g(\pi) = 2 > 0$ and $g(0) = a = 2 - \pi^3 + \pi < 0$, by the Intermediate Value Theorem, there is at least one solution in $(0, \pi)$.

3. (12 points)

(a) State the Squeeze Theorem given functions $f$, $g$ and $h$ that satisfy $f(x) \leq g(x) \leq h(x)$.

(b) Consider the following functions $f$ and $h$.

$$f(x) = -|x + 2| + 3 \quad h(x) = (x + 2)^2 + k$$

Sketch a graph of $y = f(x)$ and $y = h(x)$ for any constant $k$.

(c) The Squeeze Theorem can be used to find $\lim_{x \to a} g(x)$ for what values of $a$ and $k$?

Justify your answer.

Solution:

(a) If $f(x) \leq g(x) \leq h(x)$ when $x$ is near $a$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

(b) Here is one possible graph with $k = 3$.

![Graph of f(x) and h(x)]

(c) Here is one possible solution: Because $\lim_{x \to -2} f(x) = \lim_{x \to -2} (-|x + 2| + 3) = 3$ and $\lim_{x \to -2} h(x) = \lim_{x \to -2} ((x + 2)^2 + k) = k$, by the Squeeze Theorem, $\lim_{x \to -2} g(x) = 3$ if $a = -2$ and $k = 3$.

4. (32 points) Consider the function $f(x) = x - |3x| - 4$.

(a) State the definition of the derivative $f'(x)$.

(b) Use the definition to find $f'(x)$ for $x > 0$.

(c) Use the definition to find $f'(x)$ for $x < 0$.

(d) Is $f$ differentiable at $x = 0$? Explain.

(e) Find an equation of the line tangent to $f$ at $x = -6$.

(f) Does the derivative $f'$ have horizontal asymptotes? Justify your answer using limits.

Solution:
(a) \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \)

(b) \( f(x) = \begin{cases} 
-2x - 4, & x \geq 0 \\
4x - 4, & x < 0 
\end{cases} \)

The derivative for \( x > 0 \) is \( \lim_{h \to 0} \frac{-2(x + h) - 4 - (-2x - 4)}{h} = \lim_{h \to 0} \frac{-2h}{h} = -2 \).

(c) The derivative for \( x < 0 \) is \( \lim_{h \to 0} \frac{4(x + h) - 4 - (4x - 4)}{h} = \lim_{h \to 0} \frac{4h}{h} = 4 \).

(d) The function is continuous at \( x = 0 \); however, \( f \) is not differentiable at \( x = 0 \) because the left hand and right hand derivatives are not equal.

(e) Since \( f \) is linear at \( x = -6 \), the tangent line corresponds to \( y = f(x) : y = 4x - 4 \).

(f) Because \( \lim_{x \to \infty} f'(x) = \lim_{x \to \infty} -2 = -2 \) and \( \lim_{x \to -\infty} f'(x) = \lim_{x \to -\infty} 4 = 4 \), \( f' \) has horizontal asymptotes at \( y = -2 \) and \( y = 4 \).

5. (10 points) The graph of \( y = f(x) \) is shown below. Sketch a graph of \( y = f'(x) \). No explanation is necessary.

Solution: