1. (12 points) Match the graphs of the functions in Figure 1 to the graphs of their derivatives in Figure 2. No explanation is necessary.

Solution: (1) e (2) d (3) b (4) f

2. (20 points)

(a) Let \( y = \tan^3 x + \tan (3x) \). Find \( y'(\pi/4) \).

Solution:

\[
\begin{align*}
  y &= \tan^3 x + \tan (3x) \\
  y' &= 3 \tan^2 x \sec^2 x + 3 \sec^2(3x) \\
  y'(\pi/4) &= 3 \tan^2(\pi/4) \sec^2(\pi/4) + 3 \sec^2(3\pi/4) \\
  &= 3(1)^2(\sqrt{2})^2 + 3(-\sqrt{2})^2 = 12
\end{align*}
\]
(b) Find all values of \( \theta \) in the interval \([0, 2\pi]\) that satisfy \( 2 \cos^2 \theta = 1 + \cos \theta \).

**Solution:**

\[
2 \cos^2 \theta = 1 + \cos \theta \\
2 \cos^2 \theta - \cos \theta - 1 = 0 \\
(2 \cos \theta + 1)(\cos \theta - 1) = 0
\]

\[
\cos \theta = -\frac{1}{2} \text{ or } 1
\]

\[
\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \text{ or } 0, 2\pi
\]

3. (30 points) Let \( f(x) = \sqrt{x - 4} \).

(a) Find the domain and range of \( f \). Express your answer in interval notation.

(b) Use the definition of derivative to find \( f' \).

(c) Find the point(s) on the curve \( y = f(x) \) where the tangent line is parallel to the line \( x - 5y + 15 = 0 \).

(d) Find the value(s) of \( c \) on the interval \([4, 29]\) that satisfy the conclusion of the Mean Value Theorem for the function \( f \).

**Solution:**

(a) The domain is \([4, \infty)\). The range is \([0, \infty)\).

(b) \[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x + h - 4} - \sqrt{x - 4}}{h}
\]

Multiply by the conjugate of the numerator.

\[
= \lim_{h \to 0} \frac{\sqrt{x + h - 4} - \sqrt{x - 4}}{h} \cdot \frac{\sqrt{x + h - 4} + \sqrt{x - 4}}{\sqrt{x + h - 4} + \sqrt{x - 4}} = \lim_{h \to 0} \frac{\sqrt{x + h - 4} + \sqrt{x - 4}}{\sqrt{x + h - 4} + \sqrt{x - 4}}
\]

\[
= \lim_{h \to 0} \frac{1}{2\sqrt{x - 4}} = \frac{1}{2\sqrt{x - 4}}
\]

(c) The given line \( x - 5y + 15 = 0 \) \( \Rightarrow \) \( y = \frac{x}{5} + 3 \) has slope \( m = 1/5 \). Solve \( f'(x) = 1/5 \).

\[
\frac{1}{2\sqrt{x - 4}} = \frac{1}{5} \Rightarrow 2\sqrt{x - 4} = 5 \Rightarrow \sqrt{x - 4} = \frac{5}{2} \Rightarrow x - 4 = \frac{25}{4} \Rightarrow x = \frac{41}{4}
\]

The corresponding \( y \)-coordinate is \( y = f(41/4) = 5/2 \). The tangent line at \( \left(\frac{41}{4}, \frac{5}{2}\right) \) is parallel to the given line.
(d) The Mean Value Theorem states that given a continuous and differentiable function on \([a, b]\), there is a \(c\) in \((a, b)\) such that

\[
 f'(c) = \frac{f(b) - f(a)}{b - a}
\]

\[
 \frac{1}{2\sqrt{c - 4}} = \frac{f(29) - f(4)}{29 - 4} = \frac{5 - 0}{25} = \frac{1}{5}
\]

\[c = \frac{41}{4}\] from part (c)

4. (12 points)

\[f(t) = 12t \csc(3t)\] and \(g(t) = \begin{cases} f(t) & t \neq 0 \\ 0 & t = 0 \end{cases}\)

(a) Find \(\lim_{t \to 0} f(t)\).

(b) Is \(g\) continuous at \(t = 0\)? Justify your answer using the definition of continuity.

Solution:

(a) Use the theorem \(\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \lim_{\theta \to 0} \frac{\theta}{\sin \theta} = 1\).

\[
\lim_{t \to 0} 12t \csc(3t) = \lim_{t \to 0} \frac{12t}{\sin(3t)} = \lim_{t \to 0} 4 \cdot \frac{3t}{\sin(3t)} = 4 \cdot 1 = 4
\]

(b) By the definition of continuity, \(g\) is continuous at \(t = 0\) if \(g(0) = \lim_{t \to 0} g(t)\). Since \(g(0) = 0\) and

\[
\lim_{t \to 0} g(t) = \lim_{t \to 0} f(t) = 4,
\]

the function \(g\) is not continuous at \(t = 0\).

5. (14 points) Use implicit differentiation to find the tangent slope of \(\sin \left(\frac{x}{y}\right) = \frac{1}{y} + 1\) at \((0, -1)\).

Solution:

\[
\sin \left(\frac{x}{y}\right) = \frac{1}{y} + 1
\]

\[
\cos \left(\frac{x}{y}\right) \left(\frac{y - x \frac{dy}{dx}}{y^2}\right) = -\frac{1}{y^2} \frac{dy}{dx}
\]

We wish to find the tangent slope at \((0, -1)\).

\[
\cos(0) \left(\frac{-1 - 0}{1}\right) = -(1) \frac{dy}{dx}
\]

\[
\frac{dy}{dx} = 1
\]
6. (30 points) Let \( g(x) = \sqrt[3]{x} - \frac{1}{\sqrt[3]{x^2}} \).

(a) Find \( g' \).

(b) Find an equation of the line tangent to \( y = g(x) \) at \( x = 1 \).

(c) Find the critical numbers of \( g \).

(d) What is the linearization of \( g \) at \( a = 1 \)?

(e) Use the linearization to estimate the value of \((0.97)^{1/3} - (0.97)^{-2/3}\).

(f) Is \( g \) even, odd, or neither?

Solution:

(a) \( g(x) = \sqrt[3]{x} - \frac{1}{\sqrt[3]{x^2}} = x^{1/3} - x^{-2/3} \Rightarrow g'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} \)

(b) The tangent slope is \( g'(1) = 1/3 + 2/3 = 1 \). The \( y \)-coordinate of the point of tangency is \( g(1) = 0 \).

An equation of the line is \( y = y_0 + m(x - x_0) \Rightarrow y = 0 + 1(x - 1) \Rightarrow y = x - 1 \)

(c) The critical numbers are the values of \( x \) where \( g' = 0 \) or \( g' \) is undefined. \( g'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} \) is undefined at \( x = 0 \). Solve \( g'(x) = 0 \) to find any other critical numbers.

\[
\frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} = 0 \Rightarrow \frac{1}{3}x^{-2/3}(1 + 2x^{-1}) = 0 \Rightarrow 1 + \frac{2}{x} = 0 \Rightarrow x = -2
\]

(d) The linearization is equivalent to the equation of the tangent line at \( x = 1 \) which was found in part (b). \( L(x) = \frac{x}{x - 1} \).

(e) \((0.97)^{1/3} - (0.97)^{-2/3} = g(0.97) \approx L(0.97) = 0.97 - 1 = -0.03\)

(f) Check if \( g \) is even: \( g(-x) = g(x) \), or odd: \( g(-x) = -g(x) \).

\[
g(-x) = \sqrt[3]{-x} - \frac{1}{\sqrt[3]{(-x)^2}} = -\sqrt[3]{x} - \frac{1}{\sqrt[3]{x^2}}
\]

Since \( g(-x) \neq g(x) \) and \( g(-x) \neq -g(x) \), \( g \) is neither even nor odd.

7. (14 points) A 15-ft ladder is leaning against a wall when its base starts to slide away. By the time the base of the ladder is 12 ft from the wall, the base is moving at the rate of 2 ft/sec. How fast is the angle between the ladder and the ground changing then?

Solution:
Let \( x \) equal the distance from the base of the ladder to the wall, \( y \) equal the distance from the top of the ladder to the ground, and \( \theta \) equal the angle between the ladder and the ground. We wish to find \( d\theta/dt \) when \( x = 12 \) ft. At this moment \( dx/dt = 2 \) ft/sec, \( y = \sqrt{15^2 - 12^2} = 9 \) ft, and \( \sin \theta = 9/15 = 3/5 \).

\[
\cos \theta = \frac{x}{15} \\
-\sin \theta \frac{d\theta}{dt} = \frac{1}{15} \frac{dx}{dt} \\
3 \cdot \frac{d\theta}{dt} = \frac{1}{15} (2) \\
\frac{d\theta}{dt} = -\frac{2}{15} \cdot \frac{5}{3} = -\frac{2}{9} \text{ rad/sec}
\]

8. (18 points) No explanation is necessary for the following problems.

(a) Sketch the graph of a single function \( f \) that satisfies all of the following conditions.

- \( f \) is continuous and odd
- \( \lim_{x \to \infty} f(x) = -4 \)
- \( f'(\cdot2) \) is undefined
- \( \lim_{h \to 0} \frac{f(h) - f(0)}{h} = -4 \)

(b) Sketch the graph of a single function \( g \) that satisfies all of the following conditions.

- \( g \) is even
- \( \lim_{x \to 1^+} g(x) = \infty \)
- \( g'(0, 1) \) \( g'(x) = -3 \)
- \( g(2) = 1 \)

**Solution:** Here are two possible solutions.