1. (14 points)

(a) Differentiate the function $y = x \sin \left( \frac{1}{x} \right)$ shown below for $x \neq 0$.

**Solution:** Use the product rule and chain rule.

$$
y = x \sin \left( \frac{1}{x} \right)
$$

$$
y' = x \left[ \sin \left( \frac{1}{x} \right) \right]' + \sin \left( \frac{1}{x} \right)
$$

$$
= x \cos \left( \frac{1}{x} \right) \left( -\frac{1}{x^2} \right) + \sin \left( \frac{1}{x} \right)
$$

$$
= -\frac{1}{x} \cos \left( \frac{1}{x} \right) + \sin \left( \frac{1}{x} \right)
$$

$$
y = x \sin \left( \frac{1}{x} \right)
$$

(b) Find an equation of the normal line at $x = 1/\pi$.

**Solution:** First find the tangent slope.

$$
y' \left( \frac{1}{\pi} \right) = -\pi \cos(\pi) + \sin \pi = -\pi (-1) + 0 = \pi
$$

The normal slope equals the negative reciprocal of the tangent slope: $m = -1/\pi$. The $y$-coordinate at $x = 1/\pi$ equals

$$
y \left( \frac{1}{\pi} \right) = \frac{1}{\pi} \sin \pi = 0.
$$

The normal line is therefore

$$
y = y_0 + m(x - x_0)
$$

$$
y = -\frac{1}{\pi} \left( x - \frac{1}{\pi} \right)
$$
2. (18 points) Consider functions $f$ and $g$ whose graphs are shown below.

(a) Find $f'(-1)$, $f'(2)$, and $g'(-1)$. No explanation is necessary.
(b) Let $u(x) = g(f(x))$. Find $u'(1)$.
(c) Let $v(x) = \sqrt{3f(x) - (g(x))^2}$. Find $v'(x)$.

Solution:

(a) $f'(-1) = -1/2$, $f'(2) = \text{undefined}$, $g'(-1) = 0$

(b) 

\[
\begin{align*}
    u'(x) &= g'(f(x))f'(x) \\
    u'(1) &= g'(f(1))f'(1) \\
    &= g'(2)f'(1) = \text{undefined} \quad \text{because } g'(2) \text{ is undefined.}
\end{align*}
\]

(c) 

\[
\begin{align*}
    v(x) &= \left(3f(x) - (g(x))^2\right)^{1/2} \\
    v'(x) &= \frac{1}{2} \left(3f(x) - (g(x))^2\right)^{-1/2} \left(3f'(x) - 2g(x)g'(x)\right)
\end{align*}
\]
3. (14 points) Find equations for the horizontal tangents to the curve $y = \frac{4x}{x^2 + 1}$.

**Solution:** Use the quotient rule to find $y'$.

$$y' = \frac{(x^2 + 1)(4) - (4x)(2x)}{(x^2 + 1)^2} = \frac{4x^2 + 4 - 8x^2}{(x^2 + 1)^2} = \frac{4 - 4x^2}{(x^2 + 1)^2}$$

Solve $y' = 0$ to find the horizontal tangents.

$$4 - 4x^2 = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = -1, 1$$

The $y$-coordinates at $x = -1, 1$ are

$$y(-1) = \frac{4(-1)}{(-1)^2 + 1} = -2 \quad \text{and} \quad y(1) = \frac{4(1)}{1^2 + 1} = 2.$$

The equations for the horizontal tangents are therefore $y = -2$ and $y = 2$.

4. (12 points) The position of an object moving back and forth along a coordinate line is

$$s(t) = 2t^3 - 21t^2 + 36t, \quad 0 \leq t \leq 10.$$

(a) Find the velocity and acceleration as functions of $t$.

(b) When is the object at rest?

(c) When is the object moving forward?

**Solution:**

(a) $v(t) = s'(t) = 6t^2 - 42t + 36$, $a(t) = 12t - 42$

(b) The object is at rest when $v = 0$.

$$v(t) = 6t^2 - 42t + 36 = 0 \Rightarrow t^2 - 7t + 6 = 0 \Rightarrow (t - 1)(t - 6) = 0 \Rightarrow t = 1, 6$$

(c) The object is moving forward when $v > 0$. The graph of $v(t)$ is an upward facing parabola so $v > 0$ when $0 \leq t < 1$ and $6 < t \leq 10$. 
5. (14 points) The curves $y = x^2 + ax + b$ and $y = x^3 + c$ have the same tangent line at the point $(1, 2)$. What are the values of $a$, $b$, and $c$?

Solution: The point $(1, 2)$ lies on both curves. Substitute into the given equations.

\[
\begin{align*}
2 &= 1^2 + a(1) + b \Rightarrow a + b = 1 \\
2 &= 1^3 + c \Rightarrow c = 1
\end{align*}
\]

Find an equation for the line tangent to $y = x^3 + 1$.

\[
\begin{align*}
y' &= 3x^2 \\
y'(1) &= 3
\end{align*}
\]

The tangent line is

\[y = 2 + 3(x - 1) = 3x - 1\]

The slope of $y = x^2 + ax + b$ at $x = 1$ also must equal 3.

\[
\begin{align*}
y' &= 2x + a \\
y'(1) &= 2 + a = 3 \Rightarrow a = 1
\end{align*}
\]

Since $a + b = 1$, then $b = 0$. The two functions are $y = x^2 + x$ and $y = x^3 + 1$. 
6. (18 points) A 20-ft ladder is leaning against a wall when its base starts to slide away. By the time the base is 16 ft from the wall, the top of the ladder is sliding down at the rate of 2 ft/sec.

(a) How fast is the base of the ladder moving along the ground then?
(b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?

Solution:

(a) Let \( x \) equal the distance from the base of the ladder to the wall. Let \( y \) equal the distance from the top of the ladder to the ground. We are given that \( \frac{dy}{dt} = -2 \) ft/sec when \( x = 16 \) ft. At this moment, \( y = \sqrt{20^2 - 16^2} = 12 \) ft. We wish to find \( \frac{dx}{dt} \). Use the Pythagorean Theorem.

\[
x^2 + y^2 = 20^2
\]
\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0
\]
\[
x \frac{dx}{dt} + y \frac{dy}{dt} = 0
\]
\[
16 \frac{dx}{dt} + 12(-2) = 0
\]
\[
\frac{dx}{dt} = \frac{24}{16} = \frac{3}{2} \text{ ft/sec}
\]

(b) Let \( A \) equal the area of the triangle. We wish to find \( \frac{dA}{dt} \).

\[
A = \frac{1}{2}xy
\]
\[
\frac{dA}{dt} = \frac{1}{2} \left( x \frac{dy}{dt} + y \frac{dx}{dt} \right)
\]
\[
= \frac{1}{2} \left( 16(-2) + 12 \cdot \frac{3}{2} \right)
\]
\[
= \frac{1}{2}(-32 + 18) = -7 \text{ ft}^2/\text{sec}
\]
7. (10 points) The curve $1 + \cos y = x^4 y^2$ is shown below. Find $dy/dx$.

Solution: Use implicit differentiation.

\[
1 + \cos(y) = x^4 y^2
\]

\[
-\sin y \frac{dy}{dx} = x^4 (2y) \frac{dy}{dx} + y^2 (4x^3)
\]

\[
-\sin y \frac{dy}{dx} - 2x^4 y \frac{dy}{dx} = 4x^3 y^2
\]

\[
\frac{dy}{dx} (- \sin y - 2x^4 y) = 4x^3 y^2
\]

\[
\frac{dy}{dx} = \frac{4x^3 y^2}{- \sin y - 2x^4 y}
\]