Problem 1 Consider the functions \( f(x) = x^2 - 1 \) and \( g(x) = \sqrt{x - 1} \).

(a) Find the domains of functions \( f \) and \( g \). Give your answers in interval notation:

Domain of \( f \): This function does not have a denominator to consider, and contain a root (square or otherwise), so the domain includes all real numbers. In interval notation: \((-\infty, \infty)\).

Domain of \( g \): This function contains a square root, so we need to be sure that the portion under the square root is greater than or equal to zero:

\[
x - 1 \geq 0 \rightarrow x \geq 1.
\]

In interval notation: \([1, \infty)\).

(b) Find \( h(x) = (f \circ g)(x) \). Then give the domain of \( h \) in interval notation.

\[
h(x) = f(g(x)) = f(\sqrt{x - 1}) = (\sqrt{x - 1})^2 - 1 = x - 1 - 1 = x - 2.
\]

The domain of a composite function is the set of \( x \) for which the inner function (in this case \( g(x) \)) is defined and the set of \( x \) for which \( h(x) \) are defined. In this case, the domain for \( g(x) \) is \([1, \infty)\), and the \( x \) for which \( h(x) \) is defined \((-\infty, \infty)\), so the domain of \( h(x) \) is:

\([1, \infty)\).

(c) Which function, \( f \) or \( g \), has a greater average rate of change from \( x = 2 \) to \( x = 5 \)? Justify your answer.

The average rate of change for any function \( f \) is calculated as \( ARC = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \).

ARC for \( f \):

\[
ARC_f(x) = \frac{f(5) - f(2)}{5 - 2} = \frac{5^2 - 1 - (2^2 - 1)}{3} = \frac{24 - 3}{3} = 7.
\]

ARC for \( g \):

\[
ARC_g(x) = \frac{g(5) - g(2)}{5 - 2} = \frac{\sqrt{5 - 1} - \sqrt{2 - 1}}{3} = \frac{2 - 1}{3} = 1/3.
\]

In the calculations above, the ARC for \( f \) is greater than the ARC for \( g \), so \( f \) has the larger average rate of change between \( x = 2 \) and \( x = 5 \).
(d) Consider a function $m(x) = 2f(x)$. Prove that from $x = a$ to $x = b$, the average rate of change of function $m$ is two times the average rate of change of function $f$.

ARC for $f$:

$$\frac{f(b) - f(a)}{b - a}.$$ 

ARC for $m$:

$$\frac{m(b) - m(a)}{b - a} = \frac{2f(b) - 2f(a)}{b - a} = 2\left(\frac{f(b) - f(a)}{b - a}\right).$$

The above shows that for any two points $a$ and $b$, the average rate of change for $m(x)$ is twice that of $f(x)$. 

2
Given the equation \( f(x) = \frac{1}{2}x^{-1/2}(3x + 4)^{1/2} - \frac{3}{2}x^{1/2}(3x + 4)^{-1/2} \)

1. Find the greatest common factor of \( f(x) \).

2. Factor the equation completely.

3. Evaluate \( f(2) \).

4. Solve \( f(x) = 0 \). Is there are no solutions, state “none”.

Solutions:

1. The greatest common factor is \( \frac{1}{2}x^{-1/2}(3x + 4)^{-1/2} \).

2. Factoring the equation we have

\[
f(x) = \frac{1}{2}x^{-1/2}(3x + 4)^{-1/2}((3x + 4) - 3x) \\
= \frac{1}{2}x^{-1/2}(3x + 4)^{-1/2}(4) \\
= 2x^{-1/2}(3x + 4)^{-1/2} \\
= \frac{2}{\sqrt{x}\sqrt{3x + 4}}
\]

3. \( f(2) = \frac{2}{\sqrt{2}\sqrt{3(2) + 4}} = \frac{2}{\sqrt{2}\sqrt{10}} = \frac{1}{\sqrt{5}} \)

4. There are no solutions because \( 2 \neq 0 \).
25 points total

a) (5 pts) \( x^2 + y^2 - 8x - 6y = -21 \)

Rearrange
\( (x^2 - 8x) + (y^2 - 6y) = -21 \)

Complete the square
\[
(x^2 - 8x + 16 - 16) + (y^2 - 6y + 9 - 9) = -21
\]

\[
[(x - 4)^2 - 16] + [(y - 3)^2 - 9] = -21
\]

Rearrange
\[
(x - 4)^2 + (y - 3)^2 = 4
\]

b) (5 pts)
\[
(x - h)^2 + (y - k)^2 = r^2
\]

Center: \((h, k) = (4, 3)\)

Radius: \(r = 2\)
C) (5 pts)

Line \( x + 2y = 10 \)

Point \((4, 3)\)

\((4) + 2(3) \div 10\)

\(4 + b \div 10\)

\(10 = 10 \checkmark\)

d) (4 pts)

\[ A_\Delta = \frac{1}{2}bh \]

Find \( h \): y-int at \( x = 0 \)

\((0) + 2y = 10\)

\(y = 5\)

Find \( b \): x-int at \( y = 0 \)

\(x + 2(0) = 10\)

\(x = 10\)
\[ A_\Delta = \frac{1}{2}bh \]
\[ = \frac{1}{2} (10)(5) \]
\[ A_\Delta = 25 \]

e) (4 pts)
\[ A_0 = \pi r^2 \]
\[ A_0 = \pi (2)^2 \]
\[ = 4\pi \]
\[ A_0 = 4\pi \]

f) (2 pts)
\[ \text{Shaded Region} \]
\[ A_\Delta - \frac{1}{2} A_0 = 25 - 2\pi \]
4. (a) \( S(z) = 4\pi (z)^2 = 16\pi \text{ sq in} \)

(b) \( V(16\pi) = \frac{1}{6} \pi^{1/2} \left[ 16\pi \right]^{3/2} \)

\[ 16^{3/2} = 64 \]
\[ \frac{\pi^{3/2}}{\pi^{1/2}} = \pi^{3/2 - 1/2} = \pi \]

\[ = \frac{64\pi^{3/2}}{6\pi^{1/2}} \]

\[ = \frac{32}{3} \pi \ln^3 \]

(c) \(( V \circ S)(r) = \frac{1}{6\pi^{1/2}} \left[ 4\pi r^2 \right]^{3/2} \)

\[ 4^{3/2} = 8 \]
\[ (r^2)^{3/2} = r^3 \]

\[ = \frac{1}{6\pi^{1/2}} 8\pi^{3/2} r^3 \]

\[ = \frac{4}{3} \pi r^3 \]
4. [15 points] The volume of a sphere can be computed from its surface area \( S \) using the function \( V(S) = \frac{1}{6\pi^{1/2}} S^{3/2} \).

The surface area of a sphere in terms of its radius \( r \) is \( S(r) = 4\pi r^2 \).

(a) Find the surface area of a sphere of radius \( r = 2 \) inches.
(b) Find the volume of a sphere of radius \( r = 2 \) inches.
(c) Find and simplify \( V(S)(r) \) to obtain an expression for the volume in terms of the radius \( r \). You must show a complete solution method to this problem; you will not receive full credit for only giving an answer.

5. [20 points] These questions will be graded on your answers only.

For parts (a), (b) and (c) refer to the functions \( f \) and \( g \) in the graph below. Point A is a point on the graph of function \( g \).

(d) Which of the following is a step in the process of solving \( \frac{x+1}{x-3} \leq \frac{5x-3}{x-3} \)?

A. \( \frac{x+1}{x-3} \leq 5 \)  
B. \( 0 \leq \frac{4(x-1)}{x-3} \)  
C. \( 0 \leq 4(x-1) \)  
D. \( 0 \leq \frac{2(x-1)}{x-3} \)

(e) Which of the following is the inverse function \( f^{-1}(x) \) for \( f(x) = \frac{x-5}{3x+4} \)?

A. \( f^{-1}(x) = \frac{5+4x}{1-3x} \)  
B. \( f^{-1}(x) = \frac{3x+4}{x-5} \)  
C. \( f^{-1}(x) = \frac{4x-5}{1+3x} \)  
D. \( f^{-1}(x) = \frac{-x+5}{-3x-4} \)

(f) Which of the following is the function \( y = |2x-4| - 3 \) written as a piecewise function?

A. \( y = \begin{cases} 2x - 7 & x \geq 4 \\ -2x + 1 & x < 4 \end{cases} \)  
B. \( y = \begin{cases} 2x + 1 & x \geq 2 \\ -2x - 1 & x < 2 \end{cases} \)  
C. \( y = \begin{cases} 2x - 7 & x \geq 2 \\ -2x + 1 & x < 2 \end{cases} \)  
D. \( y = \begin{cases} 2x + 1 & x \geq \frac{1}{2} \\ -2x - 1 & x < \frac{1}{2} \end{cases} \)

(g) True or false: \( \sqrt{4-x^2} = 2-x \)

END OF EXAM

Make sure your name is at the top of the page on the front side, then put this exam page inside your bluebook and make sure it is returned with your bluebook.