Chapter 10. Momentum and Collisions

What we will learn:

• The momentum of an object is the product of its velocity and mass. It is a vector quantity and points in the same direction as the velocity vector.
• Newton’s Second Law can be phrased more generally as the force equaling the
time derivative of the momentum.

- The change of momentum, impulse, is the time integral of the force that causes the momentum change.
- In all collisions, the total momentum is conserved. The law of the conservation of momentum is the second conservation law that we encounter, after the law of energy conservation.
- Besides momentum conservation, totally elastic collisions also have the property that the total kinetic energy is conserved.
- In totally inelastic collisions, the maximum amount of kinetic energy is removed, and the colliding objects stick to each other. Total kinetic energy is not conserved, but total momentum is.
- All cases in between these two extremes are partially inelastic, and the change in kinetic energy is proportional to the square of the coefficient of restitution.
- Through the physics of collisions we can make a connection to the current physics frontier of chaos science.

10.1. Linear Momentum

When we introduced the words “force”, “position”, “velocity”, and “acceleration”, we found out that their precise physical definitions were actually quite close to their use in our everyday language. With the term “momentum” the situation is more analogous to “energy”, another term where one can make a vague connection between conversational use and precise physical meaning.

In politics, one sometimes hears that the campaign of a particular candidate gains momentum, or that legislation gains momentum in Congress. And, of course, sports teams or individual players can gain or lose momentum. What one implies with these statements is that the objects said to gain momentum are now harder to stop.

Definition

The physics definition of momentum is simply the product of an object’s mass and its velocity,

\[ \vec{p} = mv. \]  

(10.1)

As you can see, we are using the lowercase letter \( p \) as the notation for momentum. The velocity, \( \vec{v} \), is a vector; and we multiply this vector by a scalar quantity, the mass \( m \). The product is then a vector as well. The momentum vector, \( \vec{p} \), and the velocity vector, \( \vec{v} \), are parallel to each other, i.e. they point in the same direction. As a simple consequence of equation 10.1, the magnitude of the linear momentum is then given by

\[ p = mv. \]  

(10.2)

The momentum is also referred to as “linear momentum”, to distinguish it from the
angular momentum, a concept we will study in the chapter on rotation, Chapter 12.

**Momentum and Force**

Let us take the time derivative of the above definition equation (10.1). We obtain

\[
\frac{d}{dt} p = \frac{d}{dt} (mv) = m \frac{dv}{dt} + \frac{dm}{dt} v, \tag{10.3}
\]

where we have used the product rule of differentiation. For now, we will assume that the mass of the object does not change, and therefore the second term is zero. Because the time derivative of the velocity is the acceleration, we then get

\[
\frac{d}{dt} \vec{p} = m \frac{d\vec{v}}{dt} = m\vec{a} = \vec{F},
\]

according to Newton’s Second Law. The relationship

\[
\vec{F} = \frac{d}{dt} \vec{p} \tag{10.4}
\]

is then an equivalent formulation of Newton’s Second Law. This form actually is more general than the \( \vec{F} = m\vec{a} \) form, because it also holds in the cases where the mass is not constant in time. This distinction will become important when we examine rocket motion in the next chapter. Because this equation is a vector equation, we can also write it in Cartesian components:

\[
F_x = \frac{dp_x}{dt}; F_y = \frac{dp_y}{dt}; F_z = \frac{dp_z}{dt}. \tag{10.5}
\]

**Momentum and Kinetic Energy**

We have already established a relationship, \( K = \frac{1}{2} mv^2 \) (equation (8.1)), between kinetic energy, \( K \), and the speed, \( v \), and mass, \( m \). We use \( p = mv \) and get

\[
K = \frac{mv^2}{2} = \frac{m^2 v^2}{2m} = \frac{p^2}{2m}
\]

So we have the important relationship between kinetic energy, mass, and momentum:

\[
K = \frac{p^2}{2m} \tag{10.6}
\]
At this point you may ask yourself why it is useful to re-formulate much of what we have learned about velocity in terms of momentum. This reformulation is far more than idle games with mathematics. We will see that momentum is conserved in collisions; and this principle will provide an extremely helpful way to find solutions to complicated problems. But first, we need to explore the physics of changing momentum in a little more detail.

10.2. Impulse

The change in momentum is defined as the difference between the final (index \( f \)) and initial (index \( i \)) momentum:

\[
\Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i. \tag{10.7}
\]

To see why this definition is useful, we have to engage in a few lines of math. Let us start by exploring the relationship between force and momentum just a little further. We can integrate each component of the equation \( \mathbf{F} = d\mathbf{p} / dt \) over time. For the integral over \( F_x \), for example, we then obtain:

\[
\int_{t_i}^{t_f} F_x dt = \int_{t_i}^{t_f} \frac{dp_x}{dt} dt = \int_{p_{x,i}}^{p_{x,f}} dp_x = p_{x,f} - p_{x,i} \equiv \Delta p_x. \tag{10.8}
\]

This equation requires some explanation. In the second step, we have performed a substitution of variables to transform a time integration into a momentum integration. Of course, we obtain similar equations for the \( y \)- and \( z \)-components. Combining them into one vector equation then yields the following result:

\[
\int_{t_i}^{t_f} \mathbf{F} dt = \int_{t_i}^{t_f} \frac{d\mathbf{p}}{dt} dt = \int_{\mathbf{p}_i}^{\mathbf{p}_f} d\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i \equiv \Delta \mathbf{p}. \tag{10.9}
\]

The time integral of the force has a name. It is called impulse:

\[
\mathbf{J} \equiv \int_{t_i}^{t_f} \mathbf{F} dt. \tag{10.10}
\]

With this definition we now of course immediately have a relationship between the impulse and the momentum change,

\[
\mathbf{J} = \Delta \mathbf{p}. \tag{10.11}
\]

From this equation we can calculate the momentum change over some time interval, if
we know the time dependence of the force. If the force is constant or has some form that we can integrate, then we can simply evaluate the integral of equation (10.10). But formally we can define an average force

\[
\vec{F}_{av} = \frac{\int_{t_i}^{t_f} F dt}{t_f - t_i} = \frac{1}{\Delta t} \int_{t_i}^{t_f} F dt = \frac{1}{\Delta t} \int_{t_i}^{t_f} F dt
\]  

(10.12)

and then get:

\[
\vec{J} = \vec{F}_{av} \Delta t
\]  

(10.13)

You may think that this transformation trivially tells the same information as the previous equation. After all, the integration is still there, only hidden in the definition of the average force. This is true, but sometimes one is only interested in the average force. Measuring the time interval \(\Delta t\) over which the force acted as well as the resulting impulse received by an object will then tell us the average force that this object experiences during that time interval.

**Example 10.1: Baseball Home Run** (compare also example 7.3)

A major league pitcher throws a fastball that crosses the plate with a speed of 90.0 mph (=40.23 m/s) and an angle of 5.0 degrees relative to the horizontal. A batter slugs it for a home run, launching it with a speed of 110.0 mph (=49.17 m/s) at an angle of 35.0 degrees to the horizontal. The legal mass of a baseball is between 5 and 5.25 ounces. So let’s say that the mass of the baseball hit here was 5.10 ounces (= 0.145 kg).

![Figure 10.2: Baseball being hit by a bat, with initial and final momentum vectors, as well as impulse vector arrows indicated.](image.png)

**Question 1:**
What is the magnitude of the impulse the baseball receives from the bat?

**Answer 1:**
The impulse is equal to the momentum change that the baseball receives. Unfortunately, there is no shortcut, but to calculate \(\Delta \vec{v} \equiv \vec{v}_f - \vec{v}_i\) for the \(x\) and \(y\) components separately, and then add them in as vectors, and finally multiply with the mass of the baseball:
\[
\Delta v_x = (49.17 \text{ m/s})\cos 35^\circ - (40.23 \text{ m/s})\cos (180^\circ - 5^\circ) = 80.35 \text{ m/s}
\]
\[
\Delta v_y = (49.17 \text{ m/s})\sin 35^\circ - (40.23 \text{ m/s})\sin (180^\circ - 5^\circ) = 24.70 \text{ m/s}
\]
\[
\Delta v = \sqrt{\Delta v_x^2 + \Delta v_y^2} = \sqrt{80.35^2 + 24.70^2} \text{ m/s} = 84.06 \text{ m/s}
\]
\[
\Delta p = m\Delta v = (0.145 \text{ kg})(84.06 \text{ m/s}) = 12.19 \text{ kg m/s}
\]

**Possible False Answer:**
It is tempting to just add the magnitude of the initial momentum and final momentum, because they point approximately into opposite directions. This method would lead to
\[
\Delta p_{\text{wrong}} = m(v_1 + v_2) = 12.96 \text{ kg m/s}.
\]
As you can see, numerically this answer is pretty close to the correct one, only about 6\% off. It would serve as a first estimate, if you realize that the vectors almost point in opposite direction, and that in that case vector subtraction implies an addition of the two magnitudes. But if you want the right answer, you have to go through the work above.

**Question 2:**
Precise measurements show that the ball-bat contact lasts only about 1 millisecond. Suppose that for our home run the contact lasted 1.2 milliseconds. What was the magnitude of the average force exerted on the ball by the bat during that time?

**Answer 2:**
The force can be simply calculated by using the formula for the impulse,
\[
\Delta p = J = F_{av}\Delta t
\]
\[
\Rightarrow F_{av} = \frac{\Delta p}{\Delta t} = \frac{12.19 \text{ kg m/s}}{0.0012 \text{ s}} = 10200 \text{ N}
\]
This force is approximately the same as the weight of an entire baseball team! The collision of the bat and the ball results in the compression of the baseball to half of its original diameter.

Before we leave the concept of impulse, it is useful to consider it in technical applications. Some of the most important safety devices make use of the relationship between impulse, average force, and time, which we have found in equation (10.13). Airbags and seatbelts are installed in cars, and they use the principles implied in equation (10.13). If the car you are driving has a collision with another vehicle or a stationary object, then the impulse, the momentum change of your car, is rather large, and it can happen over a very short time interval. Equation (10.13) then results in a very large average force,
\[
F_{av}^c = \frac{J}{\Delta t}
\]
(10.14)
If there were no safety devices installed in your car, then your car suddenly stopping
would result in your head hitting the windshield and experiencing the impulse during a very short time of only a few milliseconds, resulting in a big average force on your head that usually causes injury or even death. Air bags and seat belts are designed to make the time over which the momentum change occurs as long as possible. Maximizing this contact time and letting the driver’s body decelerate in contact with the airbag surface minimize the force acting on the driver, greatly reducing injuries.

10.3. Conservation of Linear Momentum

Suppose we have two objects that collide with each other. The collision can be, for example, that of two billiard balls on a billiard table. This collision is called an elastic collision. Another example is the collision of a subcompact car with an 18-wheeler on a highway, where the two vehicles stick to each other. This collision is called a totally inelastic collision. In a moment we will obtain exact definitions of what is meant by the use of the terms “elastic” and “inelastic”. But first, let us look at what is happening to the momenta, \( \vec{p}_{1} \) and \( \vec{p}_{2} \), of the two colliding partners during the collision.

We find that the sum of the two momenta after the collision is the same as the sum of the two momenta before the collision (index \( i,1 \) implies initial value for particle 1, just before the collision):

\[
\vec{p}_{f,1} + \vec{p}_{f,2} = \vec{p}_{i,1} + \vec{p}_{i,2}
\]  

(10.15)

This equation is the basic expression of the law of the conservation of total momentum, the most important result of this present chapter. Let us first have a look at how to derive it and then think about its consequences.

**Derivation 10.1:**

During the collision, object 1 exerts a force on object 2. Let’s call this force \( \vec{F}_{12} \). Using our definition of the impulse and its relationship to the momentum change, we then get for the momentum change of object 2 during the collision:

\[
\int_{t_i}^{t_f} \vec{F}_{12} dt = \Delta \vec{p}_{2} = \vec{p}_{f,2} - \vec{p}_{i,2}.
\]  

(10.16)

The initial and final times are selected to contain the time of the collision process. Of course, there is also a force \( \vec{F}_{21} \) which object 2 exerts on object 1. The same argument as before now leads to:

\[
\int_{t_i}^{t_f} \vec{F}_{21} dt = \Delta \vec{p}_{1} = \vec{p}_{f,1} - \vec{p}_{i,1}.
\]  

(10.17)

Newton’s Third Law tells us that the forces are equal and opposite to each other,
\[ \vec{F}_{12} = -\vec{F}_{21}, \quad \text{or} \quad \vec{F}_{12} + \vec{F}_{21} = 0. \]  

(10.18)

Integration of this equation immediately results in:

\[ 0 = \int_{t_i}^{t_f} (\vec{F}_{21} + \vec{F}_{12}) dt = \int_{t_i}^{t_f} \vec{F}_{21} dt + \int_{t_i}^{t_f} \vec{F}_{12} dt = \vec{p}_{f,1} - \vec{p}_{i,1} + \vec{p}_{f,2} - \vec{p}_{i,2}. \]  

(10.19)

Collecting the initial momentum vectors on one side, and the final momentum vectors on the other, we then obtain the equation:

\[ \vec{p}_{f,1} + \vec{p}_{f,2} = \vec{p}_{i,1} + \vec{p}_{i,2}. \]  

q.e.d.

Equation (10.15) expresses the principle of conservation of linear momentum. The sum of the final momentum vectors is exactly equal to the sum of the initial momentum vectors. Please note that this equation does not refer to any particular conditions that the collision must follow. It is valid for all two-body collisions, elastic or inelastic. All we have used in the derivation is Newton’s Third Law.

You may object now that there also may be other, external, forces present. In the collision of billiard balls, for example, there is also the friction force due to each ball rolling or sliding across the table. Or in the collision of two cars, there is also friction between the tires and the road. But what characterizes a collision is the occurrence of very large impulses due to very large contact forces during relatively short collision times. If you integrate the external forces during these collision times, you obtain only very small or moderate impulses. Thus these external forces can usually be safely neglected in the calculation of the collision dynamics, and we can treat the two-body collisions as if there were only internal forces at work.

In addition, the same argument can be made if there are more than two partners taking part in the collision, or if there is no collision at all. As long as the net external force is zero, the total momentum will be conserved,

\[ \text{if } \sum F_{\text{ext}} = 0 \quad \text{then } \sum_{i=1}^{n} \vec{p}_i = \text{constant} \]  

(10.20)

This equation is the general formulation of the law of the conservation of momentum, the most important result of the present chapter. We will return to this general formulation again in the next chapter when we talk about systems of particles. For the remainder of the present chapter we will only consider cases in which the total net external force vanishes, and thus the total momentum is always conserved in all processes we consider.
10.4. Totally Elastic Collisions in 1 Dimension

Figure 10.3 shows the collision of two carts on an almost frictionless track. The collision was videotaped, and we show seven frames of this video, each frame six hundredths of a second apart. The cart marked with the green circle is initially at rest. The one marked with the orange square has a larger mass than the other cart and is approaching from the right. The collision happens in the frame marked with the time stamp \( t = 0.12 \text{ s} \). We can see that after the collision both carts move to the right, but the lighter cart moves with a significantly larger speed. (The speed is proportional to the horizontal distance between the markings in neighboring video frames.) We will now calculate the velocities of the carts after the collision.

![Figure 10.3: Video sequence of a collision between two carts of non-equal mass on the air track](image)

What exactly is a totally elastic collision? The answer, as in so many cases, is an idealization. In practically all collisions, at least some kinetic energy is converted into other forms of energy. This energy can be heat or the energy to deform an object, for example. But we will talk of totally elastic collisions in the limit that the total kinetic energy of the colliding objects is conserved.

This definition does not mean that each object involved in the collision retains its kinetic energy. Kinetic energy can be transferred from one object to the other, but for a totally elastic collision the sum of the kinetic energies has to remain constant:

\[
\frac{p_{1x}^2}{2m_1} + \frac{p_{1z}^2}{2m_2} = \frac{p_{1x}^2}{2m_1} + \frac{p_{1z}^2}{2m_2}
\]  

(10.21)
Because we want to restrict ourselves to collision in one dimension for now, the equation for momentum conservation (Remember: momentum is always conserved in collisions!) can be written as:

\[ p_{f,1} + p_{f,2} = p_{i,1} + p_{i,2} \]  \hfill (10.22)

Please note that this equation is still a vector equation, but we follow our previous convention that we omit vector arrows in calculations involving vectors in one dimension. Let’s look at the two equations for momentum and energy conservation. What is known, and what is unknown? Typically, we would know the two masses and initial momentum vectors, and we would want to calculate the final momentum vectors after the collision. This calculation can be done because we have two equations for two unknowns, \( p_{f,1} \) and \( p_{f,2} \). This result is by far the most common use of these equations, but it is also possible, for example, to calculate the two masses, if the initial and final momentum vectors are known. So let us go ahead and calculate the final momentum vectors. Here is what we will get:

\[
\begin{align*}
p_{f,1} &= \frac{m_1 - m_2}{m_1 + m_2} p_{i,1} + \frac{2m_1}{m_1 + m_2} p_{i,2} \\
p_{f,2} &= \frac{2m_2}{m_1 + m_2} p_{i,1} + \frac{m_2 - m_1}{m_1 + m_2} p_{i,2}
\end{align*}
\]  \hfill (10.23)

Now let’s figure out how to get to this result. The derivation is actually very instructive, because it will help you solve similar problems. So here it is:

**Derivation 10.2:**
We start with the equations for energy and momentum conservation and collect all quantities connected with object 1 on the left side, and all those connected with object 2 on the right. The equation for the (conserved) kinetic energy then becomes:

\[
\frac{p_{f,1}^2}{2m_1} - \frac{p_{i,1}^2}{2m_1} = \frac{p_{i,2}^2}{2m_2} - \frac{p_{f,2}^2}{2m_2}
\]

or

\[
m_1(p_{f,1}^2 - p_{i,1}^2) = m_2(p_{i,2}^2 - p_{f,2}^2). \]  \hfill (10.24)

For the equation of momentum conservation we obtain by rearranging

\[
p_{f,1} - p_{i,1} = p_{i,2} - p_{f,2}. \]  \hfill (10.25)

Now we divide equation (10.24) by equation (10.25) by dividing the left-hand sides of the equations by each other and the right hand sides by each other. To do this division,
we use the algebraic identity \( a^2 - b^2 = (a + b)(a - b) \). This process results in

\[
m_2(p_{i,1} + p_{f,1}) = m_1(p_{i,2} + p_{f,2}).
\] (10.26)

Now we can solve (10.25) for \( p_{f,1} = p_{i,1} + p_{i,2} - p_{f,2} \) and insert this equation into (10.26):

\[
2m_2p_{i,1} + m_2p_{i,2} - m_2p_{f,2} = m_1p_{i,2} + m_1p_{f,2}
\]

\[
p_{f,2}(m_1 + m_2) = 2m_2p_{i,1} + (m_2 - m_1)p_{i,2}
\]

\[
p_{f,2} = \frac{2m_2p_{i,1} + (m_2 - m_1)p_{i,2}}{m_1 + m_2}
\]

This result gives us one of the two desired equations above. The other equation can be obtained easily by solving (10.25) for \( p_{f,2} = p_{i,1} + p_{i,2} - p_{f,1} \) and inserting this equation into (10.26).

Perhaps it is even easier to obtain the result for \( p_{f,1} \) from the result for \( p_{f,2} \) that we just derived by exchanging the indices 1 and 2 of the two objects. It is, after all, arbitrary which object we gave the labels 1 and 2, and so the resulting equations should be symmetric under the exchange of the two labels. Use of this type of symmetry principle is very powerful and very convenient. (But it does take some getting used to at first!)

\[q.e.d.\]

With the result for the final momentum vectors in hand, we can also easily obtain expressions for the final velocities, just by using \( p = mv \). This rearrangement results in

\[
v_{f,1} = \frac{m_1 - m_2}{m_1 + m_2}v_{i,1} + \frac{2m_2}{m_1 + m_2}v_{i,2}
\]

\[
v_{f,2} = \frac{2m_1}{m_1 + m_2}v_{i,1} + \frac{m_2 - m_1}{m_1 + m_2}v_{i,2}
\] (10.27)

The two equations for the final velocities look, at first sight very similar to those for the final momentum vectors (10.23). But there is one important difference: In the second term of the right hand side of the upper equation the numerator is now \( 2m_2 \) instead of \( 2m_1 \); and conversely it is now \( 2m_1 \) instead of \( 2m_2 \) in the lower equation.

As a last point in this general section, let us calculate the relative velocity, \( v_{f,1} - v_{f,2} \), after the collision,
\( v_{f,1} - v_{f,2} = \frac{m_1 - m_2}{m_1 + m_2} v_{i,1} + \frac{2m_2 - (m_2 - m_1)}{m_1 + m_2} v_{i,2} \)
\[ = -v_{i,1} + v_{i,2} = -(v_{i,1} - v_{i,2}) \]  

(10.28)

So we see that the relative velocity just changes sign in totally elastic collisions. We will return to this result later in this chapter.

**Special Case 1: Equal masses**

If \( m_1 = m_2 = m \), the general equations (10.23) simplify considerably, because the terms proportional to \( m_1 - m_2 \) vanish, and the ratios \( 2m_1 / (m_1 + m_2) \) and \( 2m_2 / (m_1 + m_2) \) become unity. We then obtain the extremely simple result

\[
\begin{align*}
  p_{f,1} &= p_{i,2} & \text{(for special case } m_1 = m_2) \\
  p_{f,2} &= p_{i,1}
\end{align*}
\]

(10.29)

This result means that in any totally elastic collision of two objects of equal mass in one dimension the two objects simply exchange their momenta. The initial momentum of object 1 becomes the final momentum of object 2. The same is true for the velocities:

\[
\begin{align*}
  v_{f,1} &= v_{i,2} & \text{(for special case } m_1 = m_2) \\
  v_{f,2} &= v_{i,1}
\end{align*}
\]

(10.30)

**Special Case 2: One Object Initially at Rest**

Now we want to look into the case where the two masses are not necessarily the same, but where one of the two objects is initially at rest, i.e. has zero momentum. Without loss of generality we can say that object 1 is the one at rest. (Remember: the equations are invariant under exchange of the two indices 1 and 2). By using the general equations (10.23) for the momentum vectors and setting \( p_{i,1} = 0 \), we then get:

\[
\begin{align*}
  p_{f,1} &= \frac{2m_1}{m_1 + m_2} p_{i,2} & \text{(for special case } p_{i,1} = 0) \\
  p_{f,2} &= \frac{m_2 - m_1}{m_1 + m_2} p_{i,2}
\end{align*}
\]

(10.31)

In the same way we obtain for the final velocities:

\[
\begin{align*}
  v_{f,1} &= \frac{2m_2}{m_1 + m_2} v_{i,2} & \text{(for special case } p_{i,1} = 0) \\
  v_{f,2} &= \frac{m_2 - m_1}{m_1 + m_2} v_{i,2}
\end{align*}
\]

(10.32)
Suppose $v_{i,2} < 0$, i.e. the object would move from right to left, with the conventional assignment of the positive $x$-axis pointing to the right. This situation is the case in Figure 10.3. Depending on which mass is larger, we can then have three cases:

1. $m_2 > m_1 \Rightarrow (m_2 - m_1) / (m_2 + m_1) > 0$: The final velocity of object 2 still points in the same direction, but is now reduced in magnitude. This result is the case depicted in Figure 10.3.

2. $m_2 = m_1 \Rightarrow (m_2 - m_1) / (m_2 + m_1) = 0$: After the collision object 2 is now at rest, and object 1 moves with the initial velocity of object 2.

3. $m_2 < m_1 \Rightarrow (m_2 - m_1) / (m_2 + m_1) < 0$: Now object 2 bounces back; it changes direction of its velocity vector.

4. $m_2 << m_1 \Rightarrow (m_2 - m_1) / (m_2 + m_1) \approx -1$ and $2m_2 / (m_1 + m_2) \approx 0$: Object 1 would still remain at rest, and object 2 approximately reverses its velocity in the collision process. This situation occurs, for example, in the collision of a ball with the ground. In this collision object 1 is the entire Earth, and object 2 is the ball. If the collision is sufficiently elastic, the ball bounces back with the same speed it had right before the collision, but in opposite direction - up instead of down.

### 10.5 Totally Elastic Collisions in 2 or 3 Dimensions

#### Collisions with Walls

To begin our discussion of two- and three-dimensional collisions, we consider the elastic collision of an object with a solid wall. In our chapter on forces, we had seen that solid surfaces exert forces on objects that attempt to penetrate the wall. These forces are normal forces, i.e. are directed perpendicular to the surface. If this normal force acts on an object colliding with the wall, it can only give it an impulse that is perpendicular to the wall, but that has no component that is parallel to the wall.

![Figure 10.4: Elastic collision of an object with a wall.](image-url)
So we find that the momentum component of the object along the wall does not change, $p_{f,x} = p_{i,x}$. In addition, for an elastic collision, we have the condition that the kinetic energy of the object colliding with the wall has to remain the same. After all, the wall stays at rest. The kinetic energy of the object is $K = p^2 / 2m$, and so we see that $p_f^2 = p_i^2$. Because $p_f^2 = p_{f,\|}^2 + p_{f,\perp}^2$ and $p_i^2 = p_{i,\|}^2 + p_{i,\perp}^2$, we find that $p_{f,\perp}^2 = p_{i,\perp}^2$. The only two solutions are then $p_{f,\perp} = p_{i,\perp}$ and $p_{f,\perp} = -p_{i,\perp}$. Obviously, only for the second solution does the perpendicular momentum component point away from the wall after the collision. Thus it is the only physical solution.

To summarize, we find that the length of the momentum vector remains unchanged, as does the momentum component along the wall; the momentum component perpendicular to the wall changes sign, but retains the same absolute value. The angle of incidence, $\theta_i$, on the wall (compare Figure 10.4) is then also the same as the angle of reflection, $\theta_f$, 

$$\theta_i = \cos^{-1} \left( \frac{p_{i,\perp}}{p_i} \right) = \cos^{-1} \left( \frac{p_{f,\perp}}{p_f} \right) = \theta_f$$

We will see the same relationship again when we study light and its reflection off a mirror.

**Collisions of Two Objects**

We have just seen that totally elastic collisions in 1 dimension are always solvable, if we have the initial velocity or momentum conditions for the two colliding objects, as well as their masses. Again, this result is due to the fact that we have two equations for the two unknown quantities, $p_{f,1}$ and $p_{f,2}$.

For collisions in 2 dimensions, each of the final momentum vectors now has two components. So this situation leaves four unknown quantities to be determined. How many equations do we have at our disposal? Conservation of kinetic energy again provides one of them. Conservation of linear momentum provides independent equations for the $x$- and $y$-directions. Thus we have only a total of 3 equations for the 4 unknown quantities. Thus without specifying an additional boundary condition for the collision, we cannot solve for the final momenta.

For collisions in 3 dimensions the situation is even worse. Now we have two vectors with three components each, for a total of 6 unknown quantities. And we only have four equations that we can use, one from energy conservation, and three from the conservation equations for the $x$-, $y$-, and $z$-components of the momentum.

Incidentally, this fact is what makes billiard an interesting game. The final momenta after the collision are determined by where on their surface the two balls in the collision hit each other.
Speaking of billiards: we can make an interesting statement that applies here. Suppose object 2 is initially at rest, and both objects have the same mass. Then momentum conservation results in:

\[ \bar{p}_{f,1} + \bar{p}_{f,2} = \bar{p}_{i,1} \]
\[ (\bar{p}_{f,1} + \bar{p}_{f,2})^2 = (\bar{p}_{i,1})^2 \]
\[ p_{f,1}^2 + p_{f,2}^2 + 2 \bar{p}_{f,1} \cdot \bar{p}_{f,2} = p_{i,1}^2 \] (10.33)

Here we squared the equation for momentum conservation and then used the properties of the scalar product. On the other hand, conservation of kinetic energy leads to:

\[ \frac{p_{f,1}^2}{2m} + \frac{p_{f,2}^2}{2m} = \frac{p_{i,1}^2}{2m} \]
\[ p_{f,1}^2 + p_{f,2}^2 = p_{i,1}^2 \] (10.34)

for \( m_1 = m_2 \equiv m \). If we subtract this equation from the previous one, we obtain

\[ 2 \bar{p}_{f,1} \cdot \bar{p}_{f,2} = 0 \] (10.35)

But the scalar product of two vectors can only be 0 if the two vectors are perpendicular to each other, or if one of them has length 0. The latter case is in effect in a head-on collision of the two billiard balls, after which the queue ball remains at rest (\( \bar{p}_{f,1} = 0 \)) and the other ball moves on with the momentum that the queue ball had initially. In all non-central collisions both balls move after the collision; and they move in directions that are perpendicular to each other.

![Figure 10.5: Collision of two nickels.](image)

There is an experiment that you can do quite easily to see if this result of a 90-degree angle works out quantitatively. (We acknowledge Harvard’s Eric Mazur, who suggested this demonstration experiment to us). You can put two coins on a piece of paper, as
shown in Figure 10.5. Mark one of them on the paper by drawing a circle around it. Then flip the other coin with your fingers into the one that you marked (a). The coins will bounce off each other and slide briefly, before friction forces slow them down to rest (b). Then you can draw lines from the final positions of the coins back to the circle that you have drawn, as shown in (c). In (c) we also superimpose the two frames from (a) and (b) to show the motion of the coins before and after the collision, as indicated by the red arrows.

Measuring the angle between the two blue lines in Figure 10.5 results in the answer \( \theta = 80^\circ \). So our theoretically derived result of \( \theta = 90^\circ \) is not quite true for this experiment. Why?

What we have neglected is the rotation in the coins after the collision and the transfer of energy to that motion, as well as the fact that this collision is not quite perfectly elastic. However, this does not change the fact that the 90-degree rule just derived by us is a good first approximation to the problem of two colliding coins. You can perform another simple experiment of this kind on any billiard table. Again, you will find that the scattering angle is not quite 90 degrees, but that this approximation will give you a good idea on where your queue ball will move to after you hit the ball that you want to sink.

**Example 10.2: Curling**

The great Canadian sport of curling is all about collisions. One slides a 42 lb (= 19 kg) granite “stone” about 35-40 m down the ice into a target area (= concentric red, white, and blue circles with cross hairs on the ice). The teams take turns sliding stones, and the stone closest to the bull’s eye in the end wins. When a stone of the other team is the closest, the other team attempts to knock that stone out of the way.
**Question:**
Suppose that the red stone shown here had an initial velocity of 1.6 m/s in horizontal direction and got deflected to an angle of 32° relative to the horizontal, what are the two final momentum vectors right after this totally elastic collision, and what is the sum of their kinetic energies?

**Answer:**
First, let us calculate the magnitude of the initial momentum of the red stone. It is

\[ \vec{p}_{i,1} = m \vec{v}_{i,1} = (19 \text{ kg} \cdot 1.6 \text{ m/s}, 0) = (30.4, 0) \text{ kg m/s} \]

Momentum conservation dictates that the sum of the two final momenta is equal to this result. The problem specifies also that stone 1 gets deflected to +32°. According to the 90°-rule that we derived for perfectly elastic collisions, stone 2 has to be deflected to –58°. So we obtain in \( x \)-direction:

\[ 30.4 \text{ kg m/s} = p_{f,1} \cos 32° + p_{f,2} \cos(-58°) \]

and in \( y \)-direction:

\[ 0 = p_{f,1} \sin 32° + p_{f,2} \sin(-58°) \]

This problem is a system of two equations for two unknown quantities, the magnitudes of the final momenta. It may look complicated, but is really not, because the sine and cosine functions are simple numbers. We can solve the lower equation for \( p_{f,2} \) and insert it into the upper equation:

\[ 0 = p_{f,1} \sin 32° + p_{f,2} \sin(-58°) \]

\[ \Rightarrow p_{f,1} = -\frac{p_{f,2} \sin(-58°)}{\sin 32°} \]

\[ \Rightarrow 30.4 \text{ kg m/s} = (-p_{f,2} \sin(-58°)/\sin 32°) \cos 32° + p_{f,2} \cos(-58°) \]

\[ \Rightarrow p_{f,2} = \frac{30.4 \text{ kg m/s}}{\cos(-58°) - \sin(-58°) \cot 32°} = 16.1 \text{ kg m/s} \]

Inserting this result back into the \( y \)-components results in:

\[ p_{f,1} = -p_{f,2} \frac{\sin(-58°)}{\sin 32°} = 25.8 \text{ kg m/s} \]

Because we now know the angle and magnitude of each of the two momentum vectors, we have determined the momentum vectors completely. We can finish the second part of the question as to the sum of the kinetic energies of the two stones after the collision. Because this collision is totally elastic, we can simply calculate the initial kinetic energy of the red stone, because the yellow one was at rest. So our answer is:

\[ K = \frac{p_{i,1}^2}{2m} = \frac{(30.4 \text{ kg m/s})^2}{2 \cdot 19 \text{ kg}} = 24.3 \text{ J} \]
11.6. Totally Inelastic Collisions

In all collisions that are not completely elastic we cannot make use of the conservation of kinetic energy any more. These collisions are called inelastic, because some of the initial kinetic energy gets converted into internal excitation, deformation, or eventually into heat. At first sight, this loss of energy may make our task appear to be more complicated, if we want to calculate final momentum or velocity vectors of the colliding objects. However, this situation is not the case. In particular, the algebra actually becomes considerably easier when we deal with the limiting case of completely inelastic collisions.

We speak of a completely inelastic collision if the colliding objects stick to each other after the collision. This result implies that both objects have the same velocity vector after the collisions: \( \vec{v}_f,1 = \vec{v}_f,2 \equiv \vec{v}_f \). (It is obvious that in this case the relative velocity between the two colliding objects is zero.)

Using \( \vec{p} = m\vec{v} \) and momentum conservation, we then get for the final velocity vector:

\[
\vec{v}_f = \frac{m_1\vec{v}_{i,1} + m_2\vec{v}_{i,2}}{m_1 + m_2}
\] (10.36)

While this formula enables you to solve practically all problems involving totally inelastic collisions, it does not tell you how it was obtained. If you are interested in how this result was obtained, here is the (very short) derivation.

**Derivation 10.3**: We start again with the conservation law for total momentum (10.15):

\[
\vec{p}_{f,1} + \vec{p}_{f,2} = \vec{p}_{i,1} + \vec{p}_{i,2}
\]

Now we use \( \vec{p} = m\vec{v} \) and get:

\[
m_1\vec{v}_{f,1} + m_2\vec{v}_{f,2} = m_1\vec{v}_{i,1} + m_2\vec{v}_{i,2}
\] (10.37)

Having already stated that the collision is completely inelastic implies that the final velocities of the two objects are the same. This assumption results in:

\[
m_1\vec{v}_f + m_2\vec{v}_f = m_1\vec{v}_{i,1} + m_2\vec{v}_{i,2}
\]

\[
(m_1 + m_2)\vec{v}_f = m_1\vec{v}_{i,1} + m_2\vec{v}_{i,2}
\]

\[
\vec{v}_f = \frac{m_1\vec{v}_{i,1} + m_2\vec{v}_{i,2}}{m_1 + m_2}
\]

**q.e.d.**

Please note that the condition of totally inelastic collision only implies that the final velocities are the same for both objects, but that in general their momentum vectors can have quite different magnitudes.
We know from Newton’s Third Law (see chapter 3) that the forces that two objects exert on each other during a collision are equal in magnitude. But it should also be made clear that the changes in velocity, i.e. the accelerations that the two objects experience in a totally inelastic collision can be drastically different. The following example illustrates this effect.

**Example 10.3: Head-On Collision**

Consider a head-on collision of a full-size SUV, with mass \( M = 3,023 \text{ kg} \), and a compact car, with mass \( m = 1,184 \text{ kg} \). Let us assume that each had an initial speed of \( v = 50 \text{ mph} \) (=22.35 m/s), moving in opposite directions, of course. For the sake of clarity, let’s then state that the SUV initially moves with a velocity of \( -v \) and the compact car with \( +v \). If the two cars crash into each other and become entangled, we have our idealized case of a totally inelastic collision.

**Question:**
What are the velocity changes of the two cars’ velocities in the collisions?

**Answer:**
We can first calculate the final velocity that the pair has immediately after the collision. To do this calculation, we simply use the above formula and get:

\[
\begin{align*}
\nu_f &= \frac{mv - Mv}{m + M} = \frac{m - M}{m + M}v \\
&= \frac{1184 \text{ kg} - 3023 \text{ kg}}{1184 \text{ kg} + 3023 \text{ kg}}(22.35 \text{ m/s}) = -9.77 \text{ m/s}
\end{align*}
\]

So the velocity change for the SUV turns out to be:

\[
\Delta v_{\text{SUV}} = -9.77 \text{ m/s} - (-22.35 \text{ m/s}) = 12.58 \text{ m/s}
\]

But the velocity change for the compact car is:

\[
\Delta v_{\text{compact}} = -9.77 \text{ m/s} - (22.35 \text{ m/s}) = -32.12 \text{ m/s}
\]

One obtains the corresponding average accelerations by dividing by the time interval, \( \Delta t \), during which the collision takes place. This time interval is obviously the same for both cars. But this equal time interval means that the magnitude of the acceleration experienced by the body of the driver of the compact car is a factor of \( 32.12 / 12.58 = 2.55 \) bigger than that experienced by the body of the driver of the SUV.

Just from this consideration alone it is clear that it is much safer to be in the SUV in this head-on collision than in the compact car. Keep in mind that this result is true even though Newton’s Third Law teaches us that the forces exerted by the two vehicles on each other are the same (compare Example 3.3)
Ballistic Pendulum

A ballistic pendulum is a device that can be used to measure muzzle speeds of projectiles shot from firearms. The ballistic pendulum consists of a block of material into which the bullet is fired. This block of material is suspended so that it forms a pendulum. From the deflection angle of the pendulum and the known masses of bullet, \( m \), and block, \( M \), the speed of the bullet right before it hit the block can be calculated, as we will show in the following.

![Ballistic pendulum as used in an introductory physics laboratory.](image)

In order to calculate the deflection angle, we have to calculate the speed of the bullet plus block combination right after the bullet gets stuck in the block. This collision is a prototypical totally inelastic collision, and thus we can apply equation (10.36). Because the pendulum is at rest before the bullet hits it, the speed of block plus bullet is

\[
v = \frac{m}{m + M} v_b \tag{10.38}
\]

where \( v_b \) is the speed of the bullet before it hits the block, and \( v \) is the speed of the block and bullet system right after impact. The kinetic energy of the bullet was \( K_b = \frac{1}{2} m v_b^2 \), whereas the block plus bullet system has the kinetic energy

\[
K = \frac{1}{2} (m + M) v^2 = \frac{1}{2} (m + M) \left( \frac{m}{m + M} v_b \right)^2 = \frac{1}{2} m v_b^2 \frac{m}{m + M}
\]

\[
= \frac{m}{m + M} K_b
\]

Obviously, kinetic energy is not conserved in this process of the bullet getting stuck in the block, and the total kinetic energy and with it the total mechanical energy is reduced by a factor of \( m / (m + M) \). However, after the collision the block plus bullet system retains its remaining total energy in the ensuing pendulum motion, converting all of the initial kinetic energy of equation (10.39) into potential energy at the highest point,
\[ U_{\text{max}} = (m + M)gh = K = \frac{1}{2} \frac{m^2}{m + M} v_b^2 \]  

(10.40)

As usual for pendulum motion (compare chapter 6), the height \( h \) and angle \( \theta \) are related via \( h = l(1 - \cos \theta) \), where \( l \) is the length of the pendulum. Inserting this result into equation (10.40) yields then

\[
(m + M)gl (1 - \cos \theta) = \frac{1}{2} \frac{m^2}{m + M} v_b^2 \Rightarrow \\
v_b = \frac{m + M}{m} \sqrt{2gl (1 - \cos \theta)} 
\]

(10.41)

It is clear from this expression that one can measure practically any bullet speed in this way, provided one selects the mass of the block, \( M \), appropriately.

For example, if you shoot a 357 Magnum caliber round (mass \( m = 0.125 \text{ kg} \)) into a block of mass \( M = 40.0 \text{ kg} \) suspended by a 1.00 m long rope, and you get a deflection of 25.4 degrees, then equation (10.41) lets you deduce that the muzzle speed of this bullet fired from this particular gun that you used was 442 m/s (which is a typical value for this type of ammunition).

**Energy-Loss in Totally Inelastic Collisions**

Because the total kinetic energy is not conserved in totally inelastic collisions, as we have just seen, we can ask exactly how much kinetic energy is lost in the general case. This loss can be calculated by taking the difference between the total initial kinetic energy, \( K_i = \frac{p_{i1}^2}{2m_1} + \frac{p_{i2}^2}{2m_2} \), and the total final kinetic energy. We calculate the total kinetic energy for the case that the two objects stick together and move as one with the total mass of \( m_1 + m_2 \) and velocity \( \vec{v}_f \). It is:

\[
K_f = \frac{1}{2} (m_1 + m_2) \vec{v}_f^2 \\
= \frac{1}{2} (m_1 + m_2) \left( \frac{m_1 \vec{r}_{i,1} + m_2 \vec{r}_{i,2}}{m_1 + m_2} \right)^2 \\
= \frac{(m_1 \vec{v}_{i,1} + m_2 \vec{v}_{i,2})^2}{2(m_1 + m_2)} 
\]

(10.42)

Now we can take the difference of the final and initial kinetic energy and obtain for the kinetic energy loss:

\[
\Delta K = K_i - K_f = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (\vec{v}_{i,1} - \vec{v}_{i,2})^2 
\]

(10.43)

The derivation of this result involves a bit of algebra and is omitted here. What matters,
though, is that the difference in the initial velocities, i.e. the initial relative velocity, enters in the energy loss. We will return to the significance of this fact in the following section, and then again in the next chapter, when we talk about center-of-mass motion.

**Example 10.4: Forensic Science**

In the drawing in Figure 10.8, a traffic accident is sketched. The white full size pickup truck with mass $m_1=2,209$ kg is traveling north and hits the westbound blue car with mass $m_2=1,474$ kg. As the two cars smash into each other, they become entangled, i.e. stick to each other. Skid marks on the road reveal the exact location of the accident, and the direction in which the two cars were sliding immediately after the accident. This direction is measured to be $38^\circ$ relative to the direction that the pickup truck initially was moving. The white car had the right of way, because the blue car had a stop sign. The driver of the blue car, however, claimed that the driver of the white pickup truck was moving with a speed of at least 50 mph ($=22.35$ m/s), whereas the speed limit was 25 mph ($=11.18$ m/s) for these roads. Furthermore, the driver of the blue car claimed that he had stopped at the stop sign and then driven through the intersection with a speed of less that 25 mph when the white car hit him. Since the driver of the white car was speeding, he would legally forfeit the right of way and have to be declared responsible for the accident.

![Figure 10.8: Sketch of the accident scene.](image)

**Question:**
Can this version of the accident be correct?

**Answer:**
This collision is clearly a totally inelastic collision, and so we already know that the velocity of the pair of colliding cars after the collision is given by:

$$\vec{v}_f = \frac{m_1\vec{v}_{i1} + m_2\vec{v}_{i2}}{m_1 + m_2}$$
If we employ a conventional choice of coordinates, then the pickup truck only has a \( y \)-component in its velocity vector, \( \vec{v}_{i,1} = v_{i,1}\hat{y} \), where \( v_{i,1} \) is the initial speed of the pickup truck, which is what we want to calculate. The blue car only has a velocity component in the negative \( x \) direction, \( \vec{v}_{i,2} = -v_{i,2}\hat{x} \). Inserting these two facts into the equation for the final velocity leads to

\[
\vec{v}_f = \frac{-m_2 v_{i,2}}{m_1 + m_2} \hat{x} + \frac{m_1 v_{i,1}}{m_1 + m_2} \hat{y}
\]

From trigonometry, we obtain an expression for the tangent of the angle of the final velocity as the ratio of its \( y \) and \( x \) components,

\[
\tan \alpha = \frac{\frac{m_1 v_{i,1}}{m_1 + m_2}}{-\frac{m_2 v_{i,2}}{m_1 + m_2}} = -\frac{m_1 v_{i,1}}{m_2 v_{i,2}}
\]

Thus we find for the initial speed of car 1:

\[
v_{i,1} = -\frac{m_2 \tan \alpha}{m_1} v_{i,2}
\]

We have to be a bit careful with the value of the angle \( \alpha \). It is not 38 degrees, as one might conclude from a casual examination of the drawing. Instead, it is \( 90^\circ + 38^\circ = 128^\circ \), because angles are always measured relative to the positive \( x \) axis. With this result, and the known values of the masses of the two cars, we finally get:

\[
v_{i,1} = -\frac{1474 \text{ kg} \cdot \tan 128^\circ}{2209 \text{ kg}} v_{i,2} = 0.85 v_{i,2}
\]

So it follows that the pickup truck drove at a slower speed than the other car. The story of the driver of the blue car is not consistent with the facts. The driver of the pickup truck did not speed, and the driver of the blue car running the stop sign apparently caused the accident.

**Explosions**

In totally inelastic collisions two or more objects merge into one and move in unison with the same momentum after the collision. The reverse is also possible. If one object moves with initial momentum \( \vec{p}_i \) and then explodes into fragments, the process of the explosion only generates internal forces between the fragments, again according to Newton’s third law. Because an explosion takes place over a very short time, the impulse due to external
forces usually can be neglected. In this case, the total momentum is conserved. This result implies that the sum of the fragment momentum vectors has to add up to the initial momentum vector,

$$\vec{p}_i = \sum_{f=1}^{n} \vec{p}_f$$  \hspace{1cm} (10.44)

This equation relating the momentum of the exploding object just before the explosion to the sum of the fragment momentum vectors after the explosion is exactly the same as the one for a totally inelastic collision, except that the indices for the initial and final states are exchanged.

In particular, if an object breaks up into two fragments, equation (10.44) is exactly equivalent to equation (10.36), with the indices $i$ and $f$ exchanged,

$$\vec{v}_i = \frac{m_1\vec{v}_{f,1} + m_2\vec{v}_{f,2}}{m_1 + m_2}$$  \hspace{1cm} (10.45)

This relationship allows us, for example, to reconstruct the initial velocity, if the fragment velocities and masses are known.

Further, the energy release in such a two-body breakup can be calculated from equation (10.43), again with the indices $i$ and $f$ exchanged,

$$\Delta K = K_f - K_i = \frac{1}{2} \frac{m_1m_2}{m_1 + m_2} (\vec{r}_{f,1} - \vec{r}_{f,2})^2$$  \hspace{1cm} (10.46)

**Example 10.5: Particle Physics**

Use of the conservation laws of momentum and energy is essential in the work of particle physicists when they analyze the products of collisions of particles at high energies, such as the ones produced at Fermilab’s Tevatron, near Chicago, Illinois, currently the World’s highest energy proton/antiproton accelerator.

At the Tevatron particle physicists collide protons and antiprotons at total energies of 1.96 TeV (Hence the name!) Remember that $1 \text{ eV} = 1.602 \cdot 10^{-19} \text{ J}$; so $1.96 \text{ TeV} = 1.96 \cdot 10^{12} \text{ eV} = 3.2 \cdot 10^{-7} \text{ J}$. The Tevatron is set up so that the protons and antiprotons circulate in the collider ring in opposite directions with for practical purposes exactly opposite momentum vectors. The main detectors, DØ and CDF, are located at the interaction regions, where protons and antiprotons collide.

In Figure 10.9 we show an example of such a collision. In this computer-generated event display of the DØ detector and one particular collision event the proton’s initial momentum vector points exactly into the page and that of the antiproton exactly out of
Thus the total initial momentum of the proton-antiproton system is zero. The explosion produced by this collision produces several fragments, almost all of which are registered by the detector. These measurements are indicated in gray levels in the event display shown in Figure 10.9. We superimposed on this event display the momentum vectors of the corresponding particles, with their length and direction given by the information produced by the computer analysis of the detector response. On the left side of this figure, we add up the momentum vectors graphically, finding a non-zero vector sum, as indicated by the thicker green arrow.

However, momentum conservation absolutely requires that the sum of the momentum vectors of all particles produced in this collision must be zero. The conservation of momentum allows us to assign the missing momentum that would balance the momentum conservation to a particle that escaped undetected, a neutrino. With the aid of this missing momentum analysis, physicists in the DØ collaboration were able to show that the event displayed here was one in which an elusive top-quark was produced. This result is fairly recent and can be expected to lead to a Nobel Prize in the near future.

**Figure 10.9:** Event display generated by the DØ collaboration and the education office at Fermilab, showing a top-quark event. Left: momentum vectors of the detected produced particles; right: graphical addition of the momentum vectors, showing that they add up to a non-zero sum, indicated by the thicker green arrow.

### 10.7 Partially Inelastic Collisions

You can now ask what happens if a collision is neither fully elastic nor fully inelastic. Most real collisions are somewhere in between these two extremes, as we have seen in Figure 10.5 and the associated coin collision experiment. And so it is important to take a look at partially inelastic collisions in more detail. We have already seen that the relative velocity of the two collision partners in ideal elastic collisions simply changes sign, and that it becomes zero in totally inelastic collisions. In addition, we saw that the energy loss in totally inelastic collisions is proportional to the square of the relative initial velocity. So it seems logical to have a definition of partial elasticity of a collision that
involves the difference or ratio of initial and final relative velocities.

The coefficient of restitution is defined as the ratio of the magnitudes of the initial to the final relative velocities in the collision,

$$\varepsilon = \frac{|\vec{v}_{f,1} - \vec{v}_{f,2}|}{|\vec{v}_{i,1} - \vec{v}_{i,2}|}$$  \hspace{1cm} (10.47)

With this definition, we obtain a coefficient of restitution of $\varepsilon = 1$ for totally elastic collisions, and $\varepsilon = 0$ for totally inelastic collisions.

First, let us examine what happens in the limit that one of the two colliding partners is the ground (for all intents and purposes, infinitely massive) and the other one a ball. If you release the ball from some height $h_i$, we know that it reaches a speed of $v_i = \sqrt{2gh_i}$ immediately before it collides with the ground. If the collision is elastic, its speed just after the collision is the same, $v_f = v_i = \sqrt{2gh_i}$, and it bounces back to the exact same height from which it was released. If the collision is totally inelastic, as is the case for a ball of putty that falls to the ground and then just stays there, then the final speed is 0. For all cases in between, one can measure the coefficient of restitution from the height $h_f$ that the ball returns to:

$$h_f = \frac{v_f^2}{2g} = \frac{\varepsilon^2 v_i^2}{2g} = \varepsilon^2 h_i$$

$$\Rightarrow \varepsilon = \sqrt{\frac{h_f}{h_i}}$$  \hspace{1cm} (10.48)

Using this method to measure the coefficient of restitution, one finds for baseballs that $\varepsilon = 0.58$, for example, for typical relative velocities that would be involved in ball-bat collisions in major-league games.

In general, we can then calculate the kinetic energy loss in partially inelastic collisions as:

$$\Delta K = K_i - K_f = \frac{1}{2} \left( \frac{m_1 m_2}{m_1 + m_2} \right) \varepsilon^2 (v_{i,1} - v_{i,2})^2$$  \hspace{1cm} (10.49)

**Partially Inelastic Collision with a Wall**

If you play racquetball or much more so in squash, you know that the ball loses energy when you hit it against the wall. While we have seen above that the angle with which a ball bounces off the wall in an elastic collision is the same as the angle that it hit the wall with, the answer of the final angle is not so clear for the present partially elastic case.

The key to obtaining a first approximation to this situation is to consider only the normal force, which acts in perpendicular direction to the wall. Then momentum component
along the wall still remains unchanged, just like in the totally elastic case. But now the momentum component perpendicular to the wall does not simply get inverted, but also reduced in magnitude by the coefficient of restitution, $p_{f,\perp} = -\epsilon p_{i,\perp}$. In this approximation we obtain an angle of reflection

$$
\theta_f = \arctan \frac{p_{f,\perp}}{p_{f,p}} = \arctan \frac{\epsilon p_{i,\perp}}{p_{i,p}} < \theta_i
$$

(10.50)

The magnitude of the final momentum vector is also reduced

$$
p_f = \sqrt{p_{f,p}^2 + p_{f,\perp}^2} = \sqrt{p_{i,p}^2 + \epsilon^2 p_{i,\perp}^2} < p_i
$$

(10.51)

If we want a more quantitative description, we also need to include the effect of a friction force between ball and wall, acting for the duration of the collision. (This is why squash balls and racquetballs leave blue marks on the walls.) Further, the collision with the wall also changes the rotation of the ball, and thus additionally alters the direction and kinetic energy of the ball bouncing off. But equation (10.50) and (10.51) still provide a very reasonable first approximation to partially inelastic collisions with walls.

![Figure 10.10: Partially inelastic collision of a ball with a wall.](image)

### 10.8. Billiards and Chaos

Let us look at billiard systems in an abstract way. Our “billiard” is going to be a rectangular (or even square) area, in which particles can bounce around and have elastic collisions with the wall. Between collisions, these particles move on straight trajectories without energy loss. When we start two particles close to each other, as in the left part of the drawing in Figure 10.11, they stay close to each other. In the left panel, we show the trajectories (red and green lines) of two particles, which started close to each other and with the same initial momentum (indicated by the red arrow). And you can clearly see that the two particles stay close the entire time.
The situation becomes qualitatively different when you add a circular wall in the middle of the billiard. Now each collision with the circle drives the two trajectories farther apart. In the right panel, you can see that one collision with the circle was enough to separate the red and green line for good. This type of billiard is called a Sinai-billiard, named after the Russian Academician Yakov Sinai (1935- ) who studied it first in 1970. While the conventional billiard system shows regular motion, the Sinai-billiard exhibits chaotic motion. And surprisingly these billiard systems are still not fully explored. Cutting edge modern physics research gains new knowledge of these systems all the time, and thus explores the physics of chaos.

One example: Only in the last decade have we begun to understand the decay properties of these systems. If you cut a small hole into the wall of a conventional billiard and measure the time it takes a particle to hit this hole and escape, you obtain a power-law decay time distribution. If you do the same for the Sinai-billiard, you obtain an exponential time dependence of the escape.

These types of investigations are by no means idle theoretical speculations. If you want, you can do the following experiment. Place a billiard ball on the surface of a table and hold on to it. Then hold a second one as exactly as you can over the first one and release it from a height of a few inches (or centimeters). You will see that the upper one cannot be made to bounce on the lower one more than three or four times, before falling off in some uncontrollable direction. Even if you could fix the location of the two balls to atomic precision, the upper ball would fall off after only ten to fifteen bounces. This result means that after only a few collisions your ability to predict the outcome of this experiment has completely vanished. This limitation of predictability goes to the heart of chaos science. It is one of the main reasons, for example, that exact long-time weather forecasting is impossible. Air molecules, after all, bounce off each other, too, and we have just examined how unpredictable these scattering become after only short time spans.
Laplace’s Demon

Marquis Pierre-Simon Laplace (1749-1827) was an eminent French physicist and mathematician of the 18th century. He lived during the time of the French Revolution and great societal upheavals, characterized by the struggle for self-determination and freedom. No painting symbolizes this better than “Liberty Leading the People” (1830) by Eugène Delacroix, shown here.

![Image of Liberty Leading the People](image)

Figure 10.12: “Liberty Leading the People”, Eugène Delacroix (Louvre, Paris, France).

Laplace had an interesting idea, now known as Laplace’s Demon. He reasoned that everything is made out of atoms, and all atoms obey differential equations governed by the forces acting on them. If one were to feed all initial positions and velocities of all atoms, together with all force laws, into a huge computer (he called this an “intellect”), then “for such an intellect nothing could be uncertain and the future just like the past would be present before its eyes”. This situation implies that everything is predetermined, and we are all only cogs in a huge clockwork. If you think this line of reasoning through to its ultimate consequence, it means that nobody has free will. And Laplace came up with this idea in a period when quite a few people believed that they could achieve free will, if they only killed enough people presently in power.

The rescue from Laplace’s Demon comes from the combination of two areas of physics. One is chaos science that tells us that long-term predictability is sensitively dependent on the knowledge of the initial conditions, as seen in the example of bouncing billiard balls above. And of course the same principle applies to molecules bouncing off each other, as for example air molecules do. The other ingredient is the impossibility to specify both the position and the momentum of any object exactly at the same time. We will return to this point when we discuss the Uncertainty Relation in quantum physics (chapter 33). But what we can already take from this discussion is the certainty that the concept of free will is still alive and well – the concept of long-term predictability of large systems like the weather or the human brain is impossible. The combination of chaos theory and quantum
theory ensure that Laplace’s Demon or any computer cannot possibly calculate and predict what your individual decisions will turn out to be.

**What we have learned/Exam Study Guide:**

- Momentum is defined to be the product of mass times velocity: \( \vec{p} = m \vec{v} \).
- Newton’s Second Law: \( \vec{F} = \frac{d\vec{p}}{dt} \).
- The impulse is the momentum change and is equal to the integral over the applied external force: \( \vec{J} = \Delta \vec{p} = \int_{t_i}^{t_f} \vec{F} dt \).
- In collisions of two objects, momentum can be exchanged, but the sum of the momenta of the colliding objects remains constant, \( \vec{p}_{f,1} + \vec{p}_{f,2} = \vec{p}_{i,1} + \vec{p}_{i,2} \).
- We distinguish between totally elastic, totally inelastic, and partially elastic collisions.
- In totally elastic collisions, the total kinetic energy also remains constant,
  \[
  \frac{p_{f,1}^2}{2m_1} + \frac{p_{f,2}^2}{2m_2} = \frac{p_{i,1}^2}{2m_1} + \frac{p_{i,2}^2}{2m_2}.
  \]
- One-dimensional totally elastic collisions can be solved in general, and the final velocities of the two colliding objects can be expressed as a function of the initial velocities:
  \[
  v_{f,1} = \frac{m_1 - m_2}{m_1 + m_2} v_{i,1} + \frac{2m_2}{m_1 + m_2} v_{i,2}
  \]
  \[
  v_{f,2} = \frac{2m_1}{m_1 + m_2} v_{i,1} + \frac{m_2 - m_1}{m_1 + m_2} v_{i,2}
  \]
- In totally inelastic collisions, the colliding partners stick together after the collision and have the same velocity, \( \vec{v}_f = (m_1 \vec{v}_{i,1} + m_2 \vec{v}_{i,2}) / (m_1 + m_2) \).
- All partially inelastic collisions between the two extremes are characterized by a coefficient of restitution, defined as the ratio of the magnitudes of the final and the initial relative velocity, \( \varepsilon = |\vec{v}_{f,1} - \vec{v}_{f,2}| / |\vec{v}_{i,1} - \vec{v}_{i,2}| \). The kinetic energy loss in partially inelastic collisions is then given by
  \[
  \Delta K = K_i - K_f = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \varepsilon^2 (\vec{v}_{i,1} - \vec{v}_{i,2})^2.
  \]