MCEN 5023/ASEN 5012

Chapter 9

Constitutive Equations - Plasticity

Fall, 2006

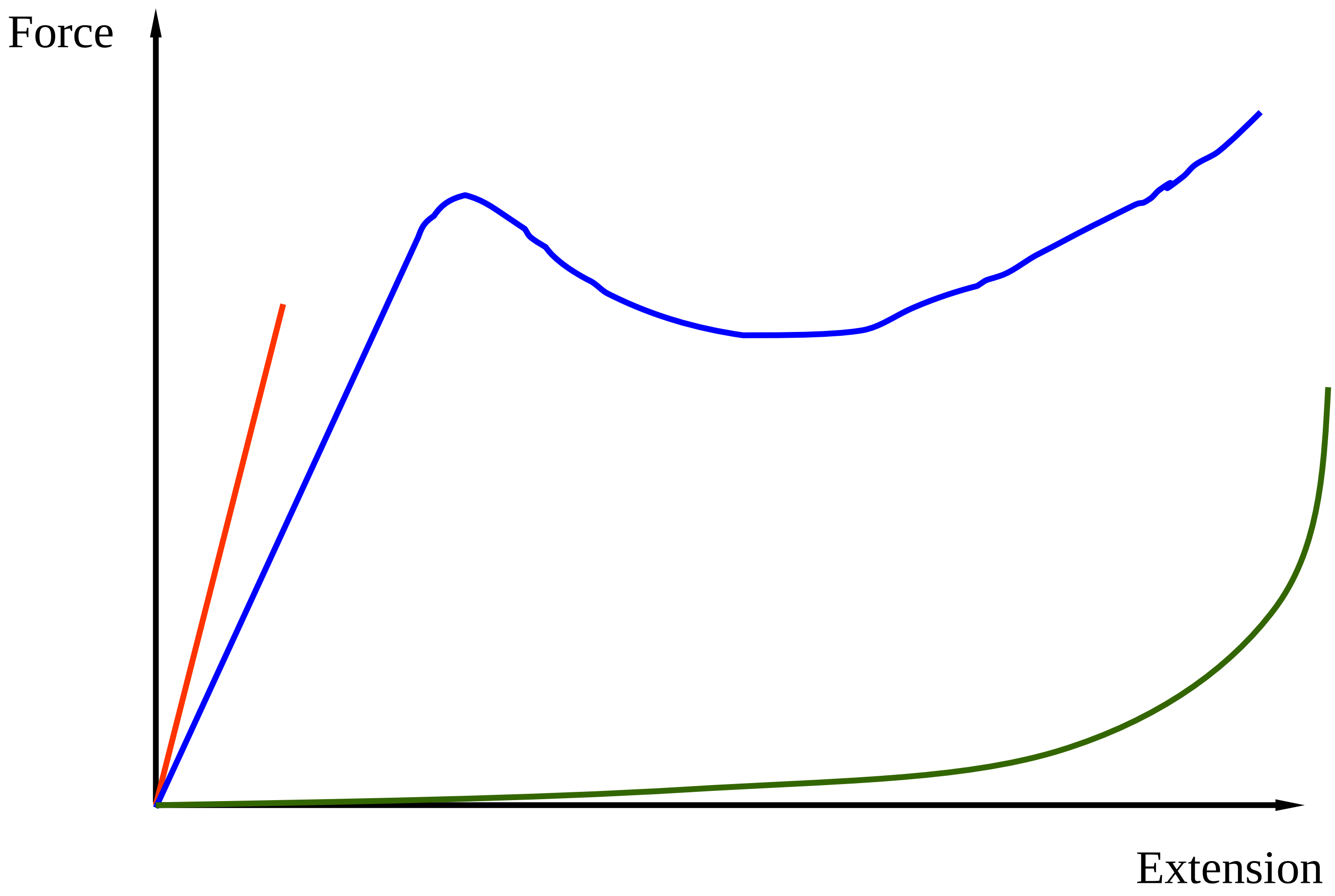
Constitutive Equations

Mechanical Properties of Materials:

- Modulus of Elasticity
- Tensile strength
- Yield Strength
- Compressive strength
- Hardness
- Impact strength
- Creep

Constitutive Equations

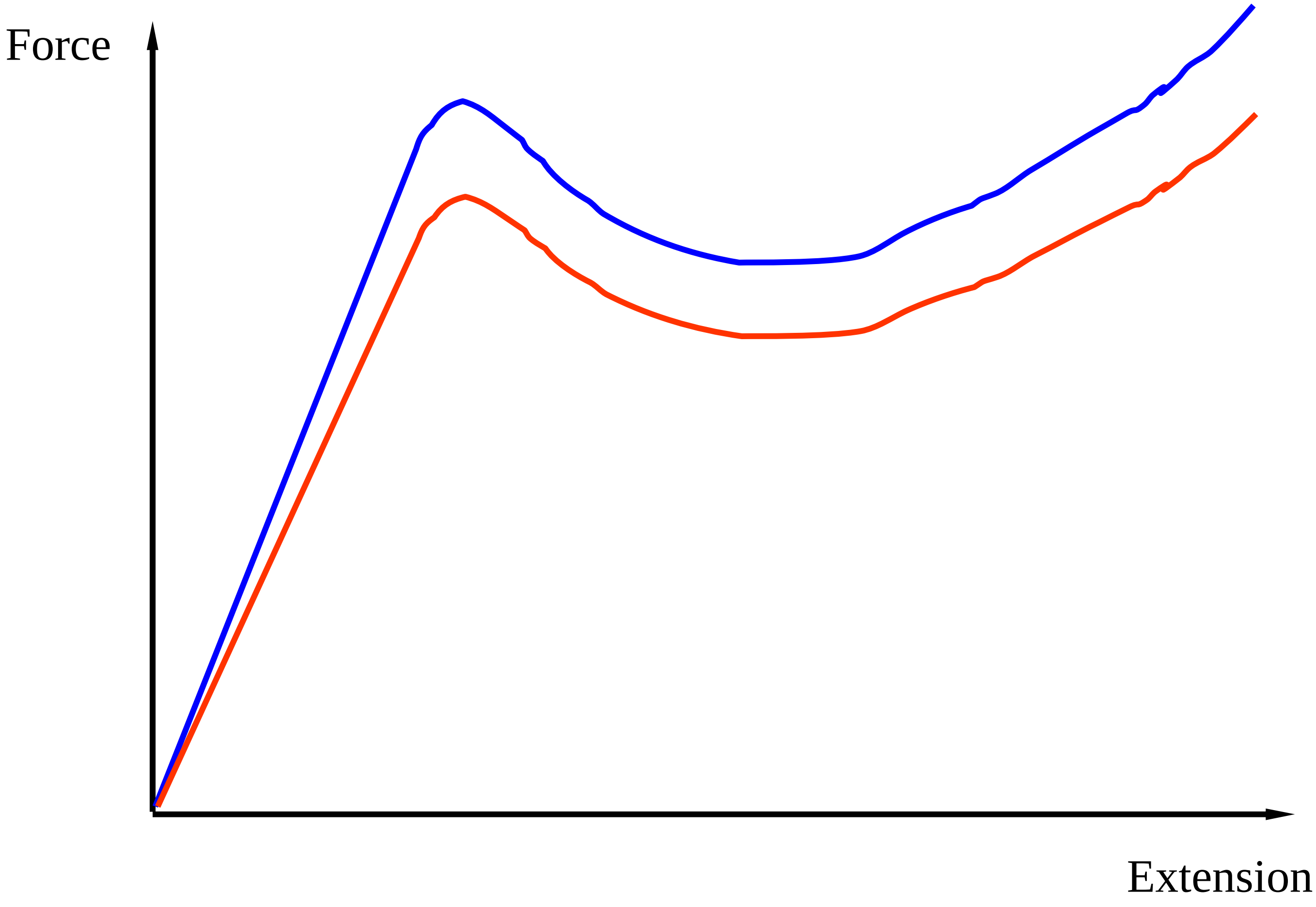
Mechanical Properties of Materials:



Constitutive Equations

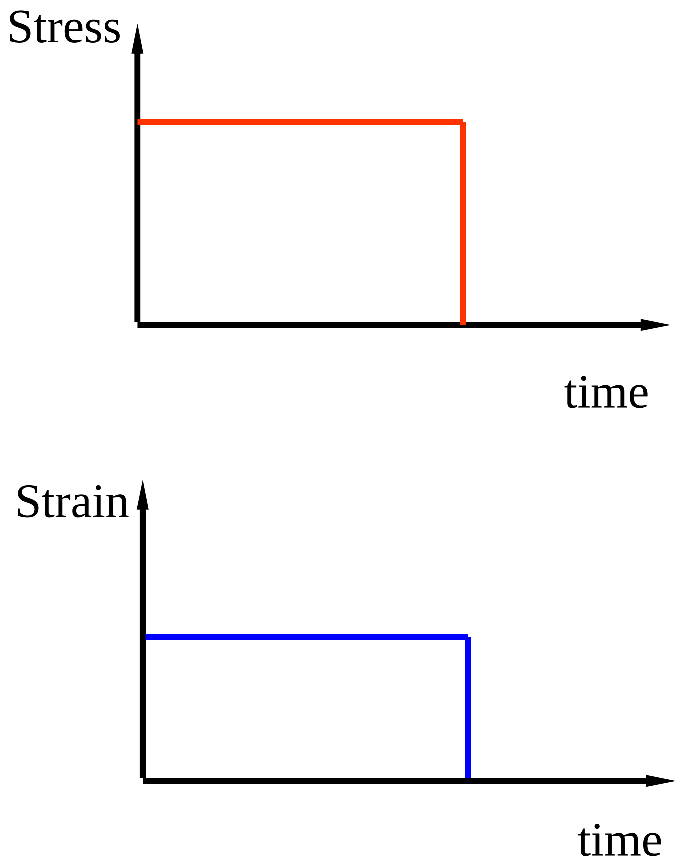
Mechanical Properties of Materials:

Time dependent, rate dependent

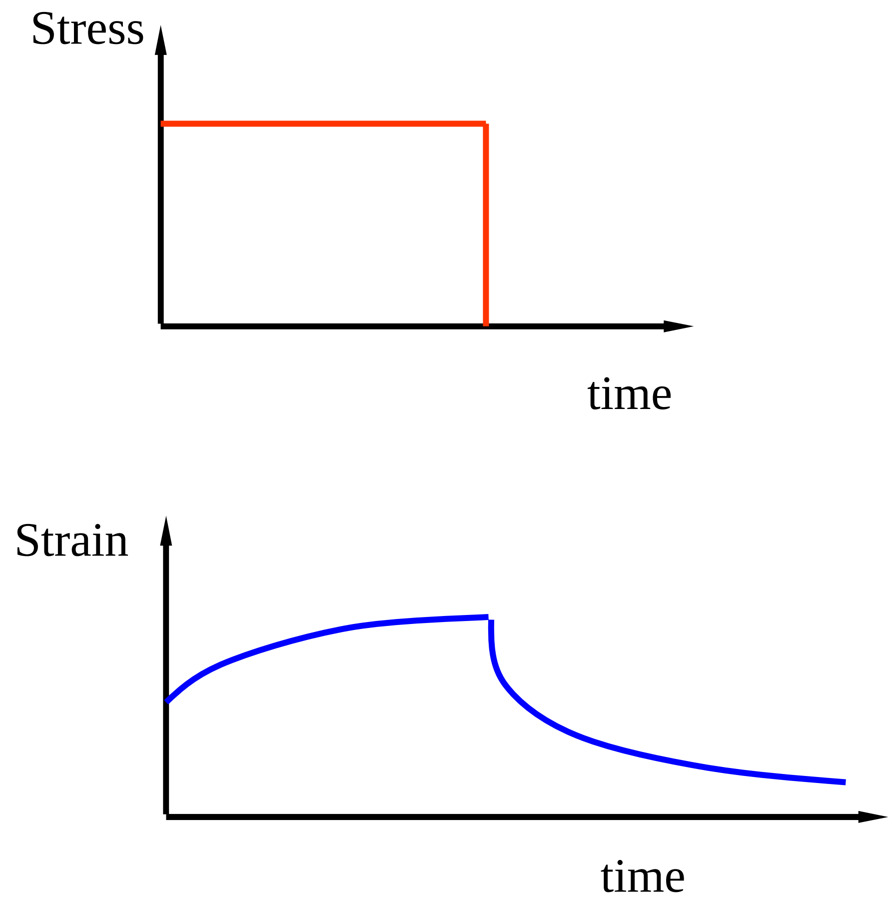


Constitutive Equations

Mechanical Properties of Materials:



Elasticity

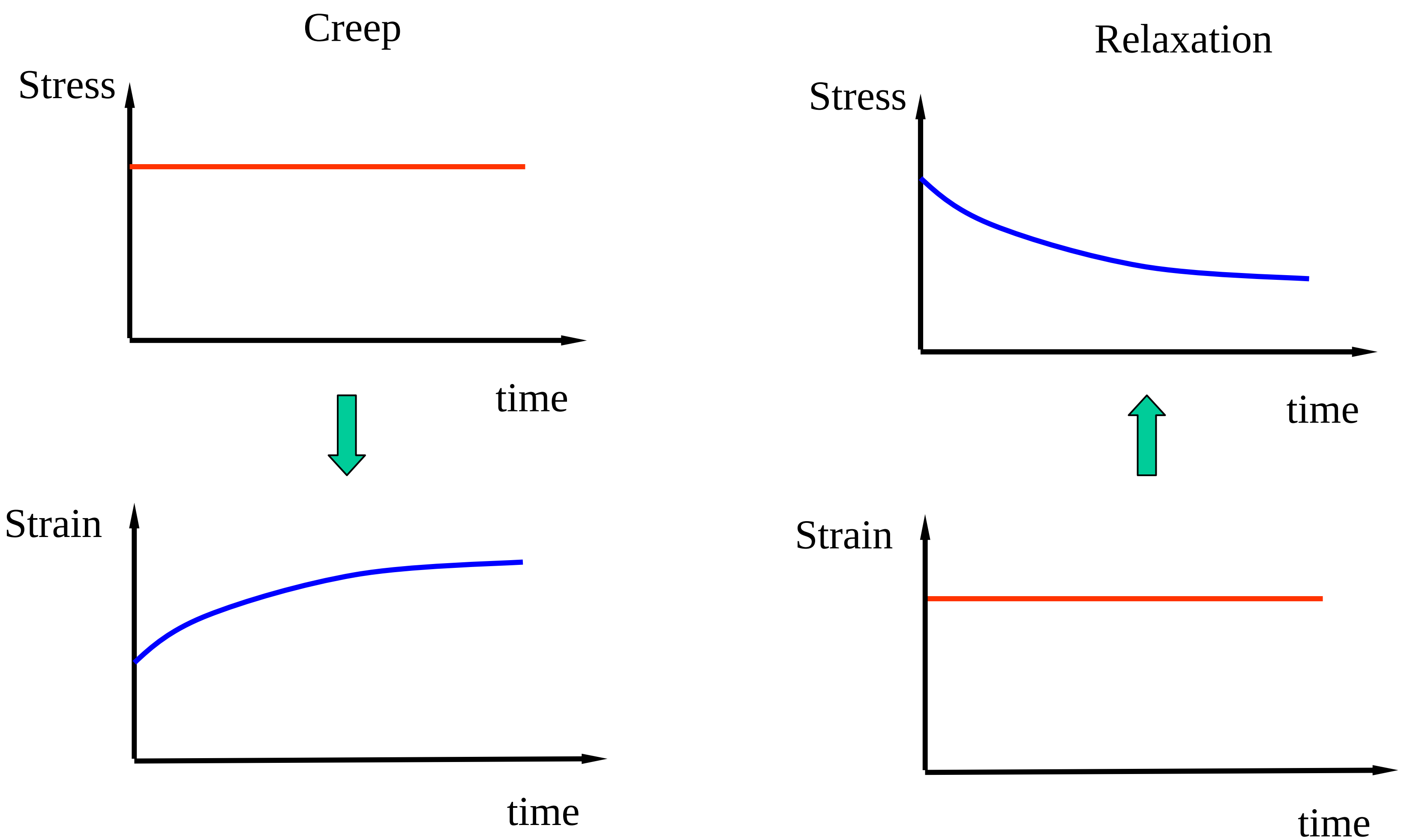


Viscoelasticity

Constitutive Equations

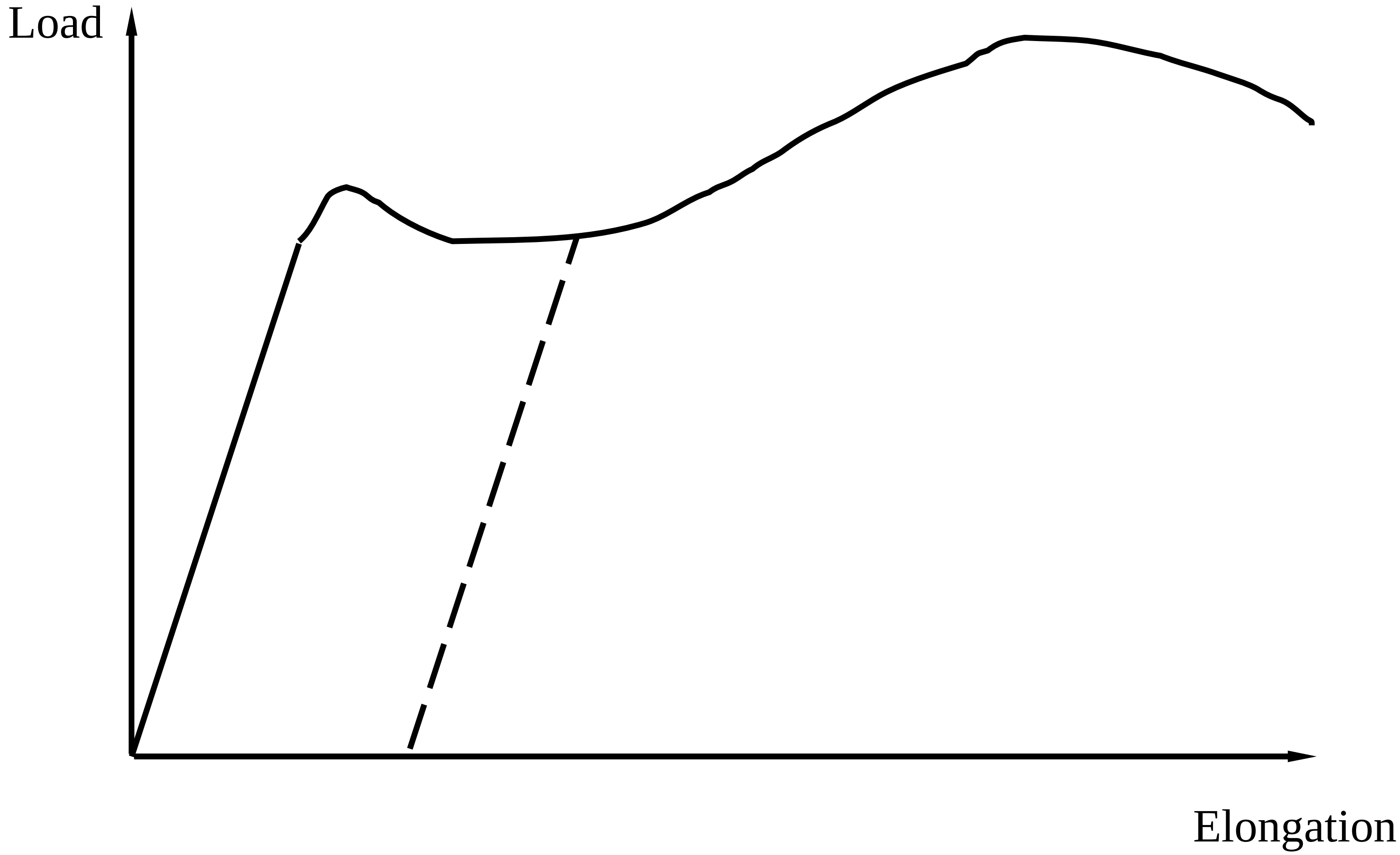
Mechanical Properties of Materials:

Viscoelasticity



Constitutive Equations

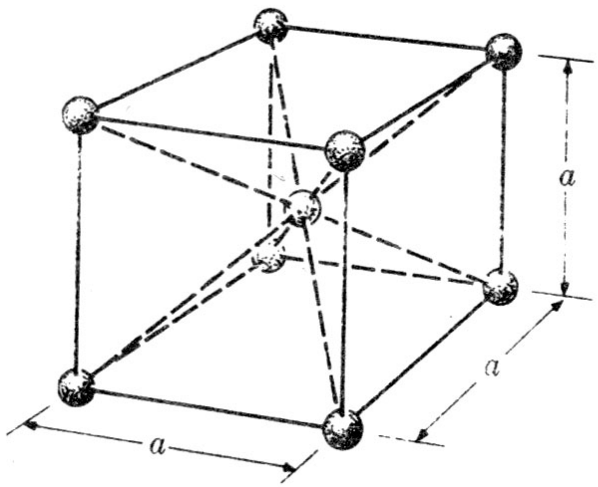
Plastic Deformation in Materials



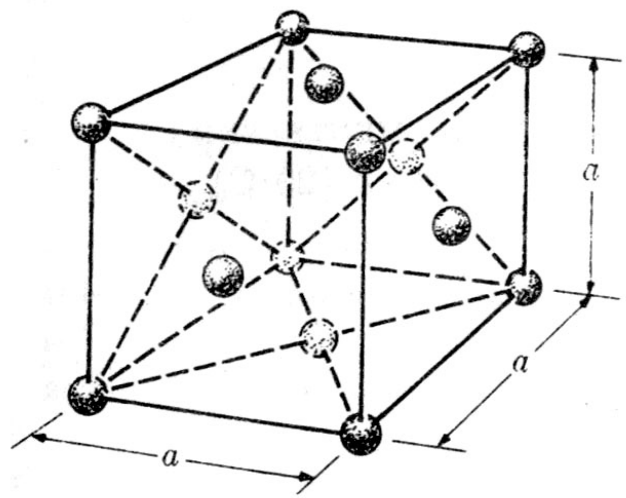
Constitutive Equations

Physics of plasticity of metals

Some typical crystal structure of metals



Body Centered Cubic
(b.c.c)



Face Centered Cubic
(f.c.c)

Hexagonal close-packed
(h.c.p)

Constitutive Equations

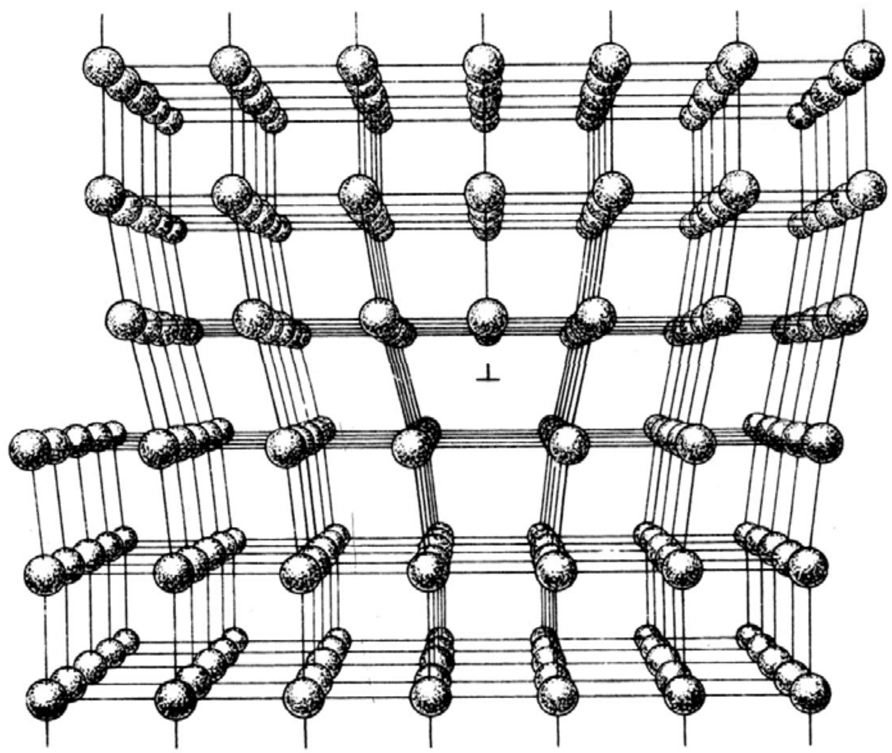
Physics of plasticity of metal

Crystal Structure of Some Common Elements and Lattice Dimensions at Room Temperature (20°C)

Element	Type of structure	Atoms per unit cell	Lattice constants, A† a	Lattice constants, A† c	Distance of closest approach, A
Aluminum	f.c.c.	4	4.0490		2.862
Beryllium	h.c.p	2	2.2854	3.5841	2.225
Cadmium	h.c.p.	2	2.9787	5.617	2.979
Carbon (diamond)	diamond cubic	8	3.568		1.544
Chromium	b.c.c.	2	2.8845		2.498
Cobalt α	h.c.p.	2	2.507	4.069	2.506
Cobalt β	f.c.c.	4	3.552		2.511
Copper	f.c.c.	4	3.6153		2.556
Germanium	diamond cubic	8	5.658		2.450
Gold	f.c.c.	4	4.0783		2.884
Iron α	b.c.c.	2	2.8664		2.481
Iron γ (extrapolated)	f.c.c.	4	3.571		2.525
Lead	f.c.c.	4	4.9495		3.499
Lithium	b.c.c.	2	3.5089		3.039
Magnesium	h.c.p.	2	3.2092	5.2103	3.196
Manganese α	cubic	58	8.912		2.24
Manganese β	cubic	20	6.300		2.373
Molybdenum	b.c.c.	2	3.1466		2.725
Nickel	f.c.c.	4	3.5238		2.491
Platinum	f.c.c.	4	3.9237		2.775
Potassium	b.c.c.	2	5.344		4.627
Silicon	diamond cubic	8	5.4282		3.351
Silver	f.c.c.	4	4.0856		2.888
Sodium	b.c.c.	2	4.2906		3.715
Tantalum	b.c.c.	2	3.3026		2.860
Tin α	diamond cubic	8	6.47		2.81
Titanium α	h.c.p.	2	2.9504	4.6833	2.89
Tungsten α	b.c.c.	2	3.1648		2.739
Zinc	h.c.p.	2	2.664	4.945	2.664

Constitutive Equations

Physics of plasticity of metals- Defects

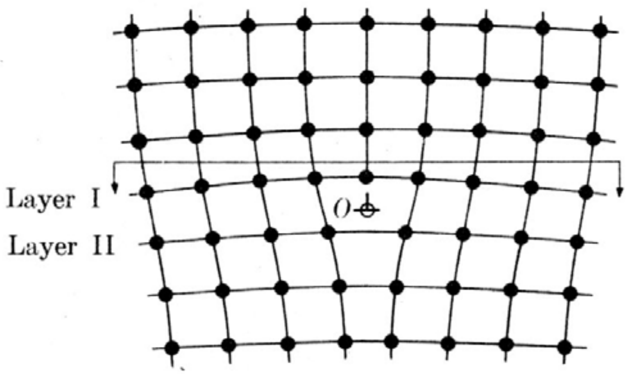


Line defects

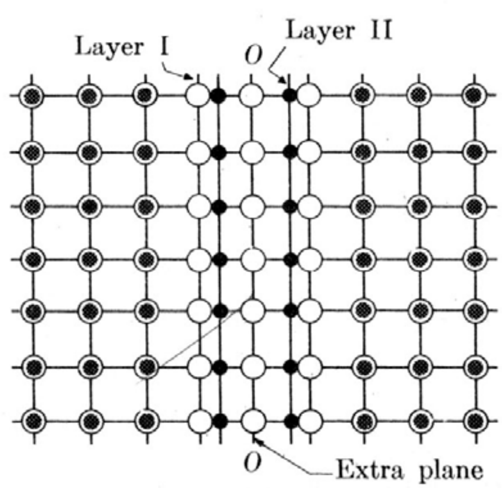
Constitutive Equations

Physics of plasticity of metals- Dislocations

Edge dislocation

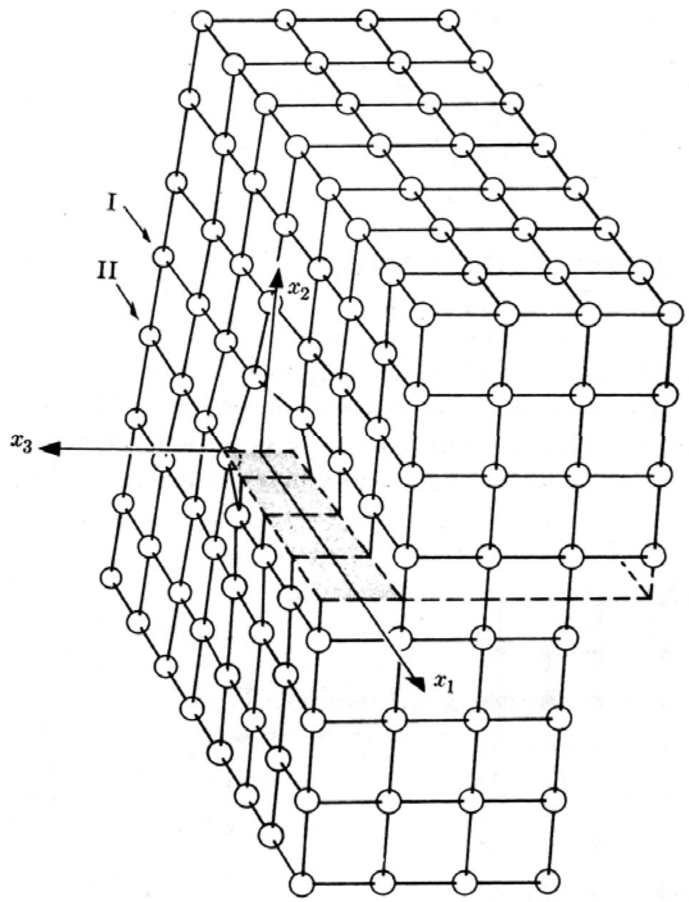


(b)



(a)

Screw dislocation



Constitutive Equations

Physics of plasticity of metals
- Dislocations

From edge dislocation
to screw dislocations

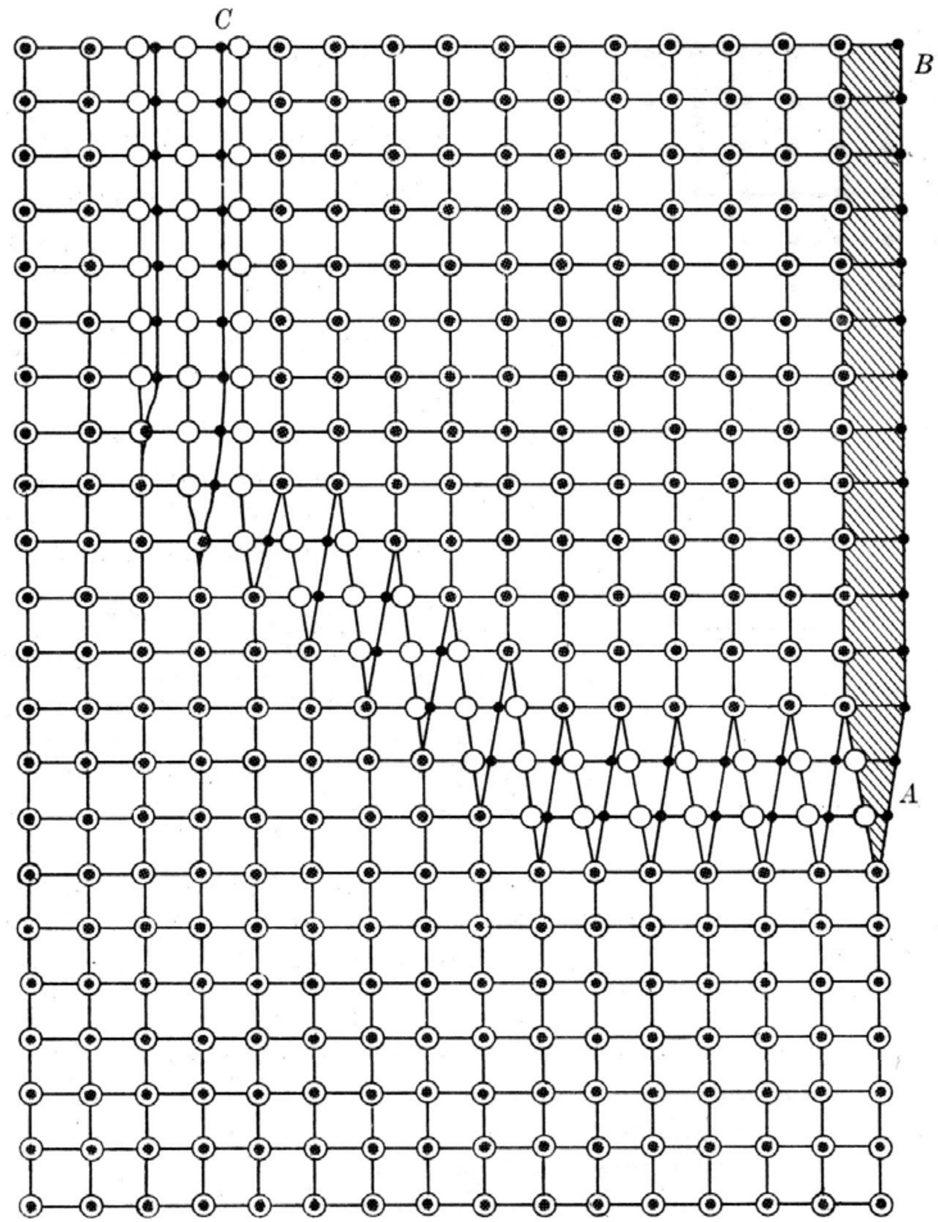
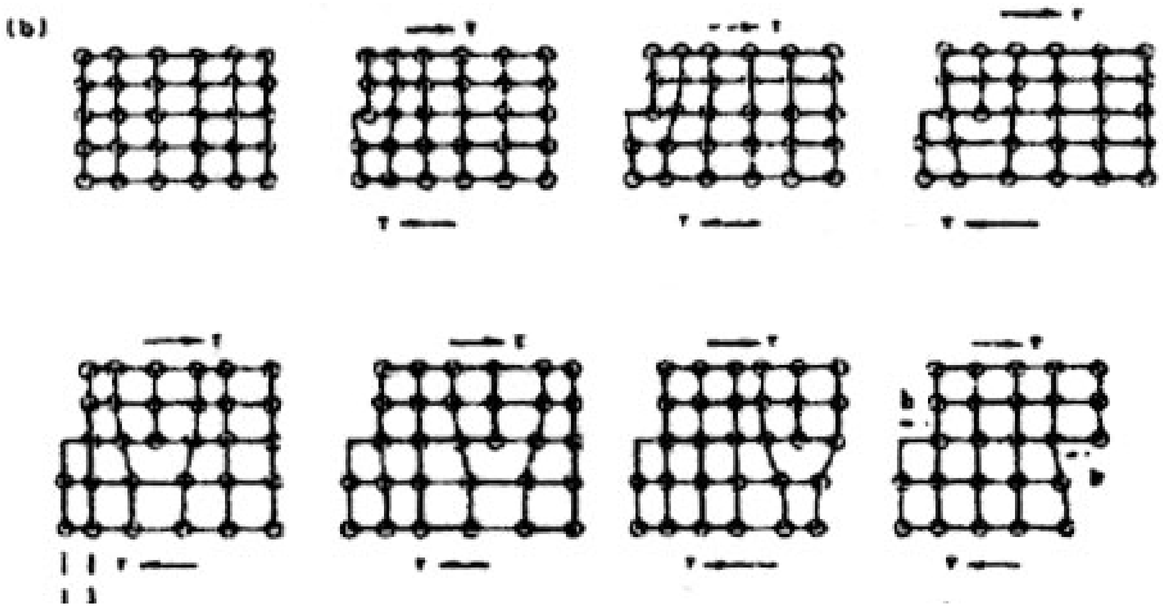


Fig. 4.3. Arrangement of atoms in a curved dislocation. Open circles represent the atomic plane just above the slip plane; smaller circles represent the atoms just below. The segment at C is pure edge dislocation; the segment at A is pure screw dislocation; the region in between is part edge and part screw dislocation.

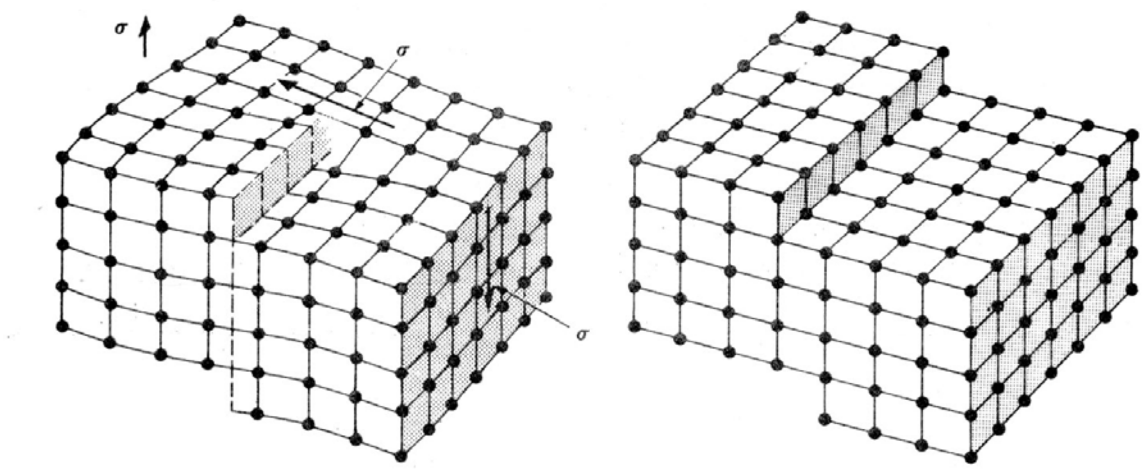
Constitutive Equations

Physics of plasticity of metals- Motion of dislocations

Edge dislocations



Screw dislocations



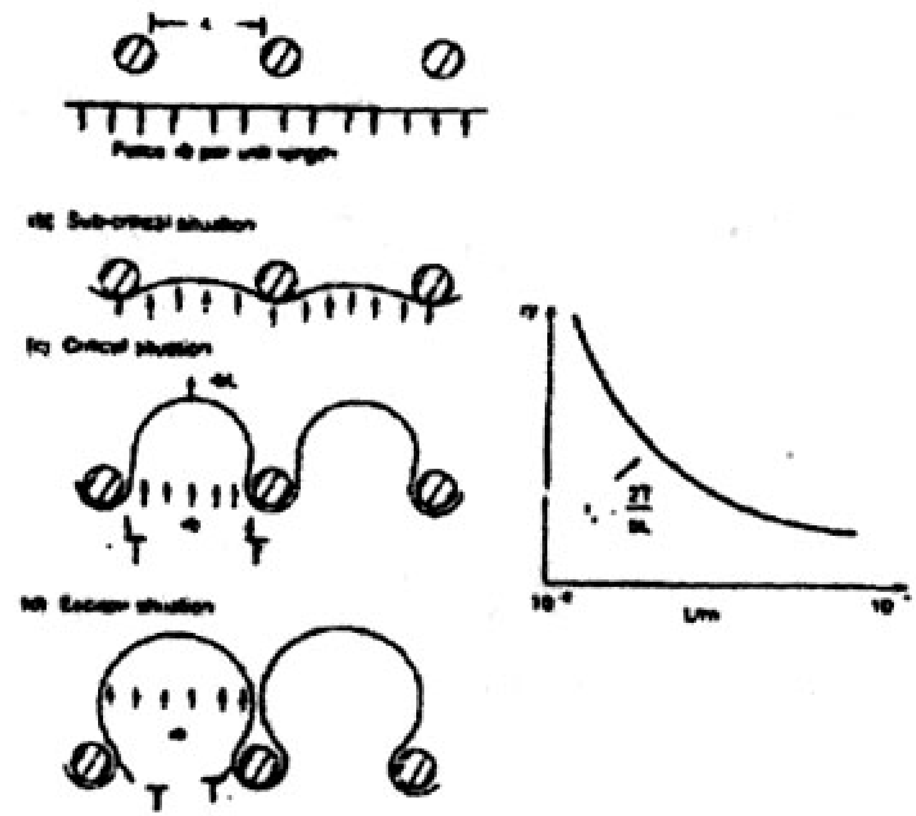
Constitutive Equations

Physics of plasticity of metals- Motion of dislocations

Constitutive Equations

Physics of plasticity of metals- Hardening

Hardening is due to obstacles to the motion of dislocations; obstacles can be particles, precipitations, grain boundaries.



Constitutive Equations

Physics of plasticity of metals- Hardening

Three types of hardening mechanism

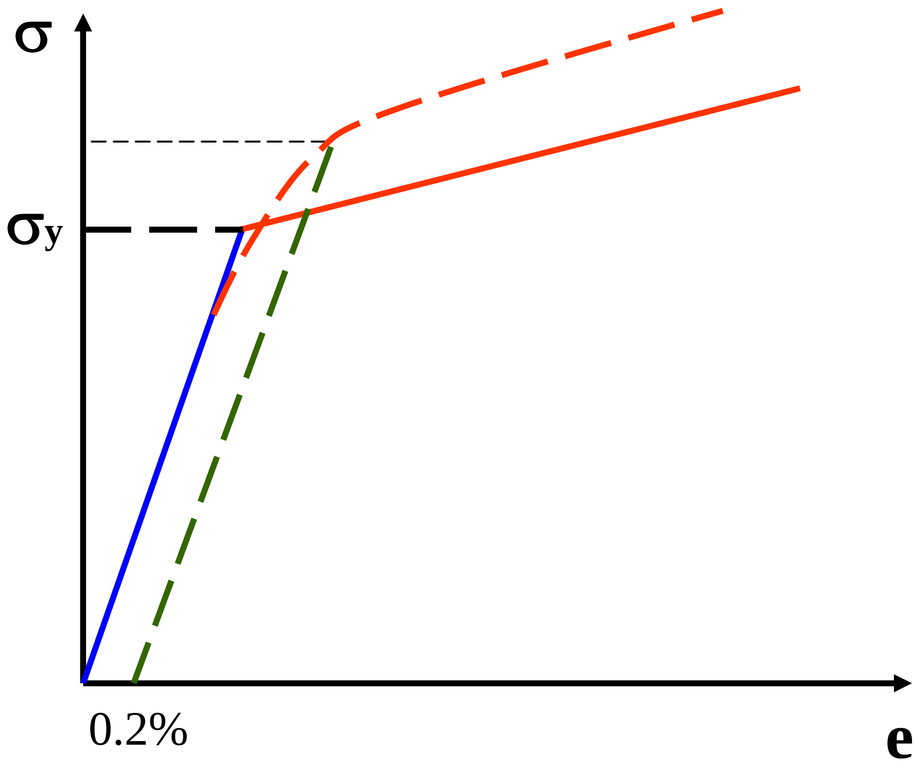
Solid solution hardening

Precipitation hardening

Strain hardening

Constitutive Equations

Physics of plasticity of metals- Yield



A simple tension test

σ_y is yield strength

Question: how can relate the σ_y from 1D test to yield in a general 3D stress state?

Constitutive Equations

Physics of plasticity of metals- Yield Criteria

It has been found experimentally

1.Tresca yield condition (1864)

$$\tau_{\max} = \frac{1}{2}\left|\sigma_1 - \sigma_3\right| \le \tau_y$$

τ_y Shear yield strength

2.Mises yield condition (1913)

$$\bar{\sigma} < \sigma_y$$

$\bar{\sigma}$ Mises stress Equivalent tensile stress

Constitutive Equations

Physics of plasticity of metals- Yield Criteria

Mises yield condition

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 \leq 2\sigma_y^2$$

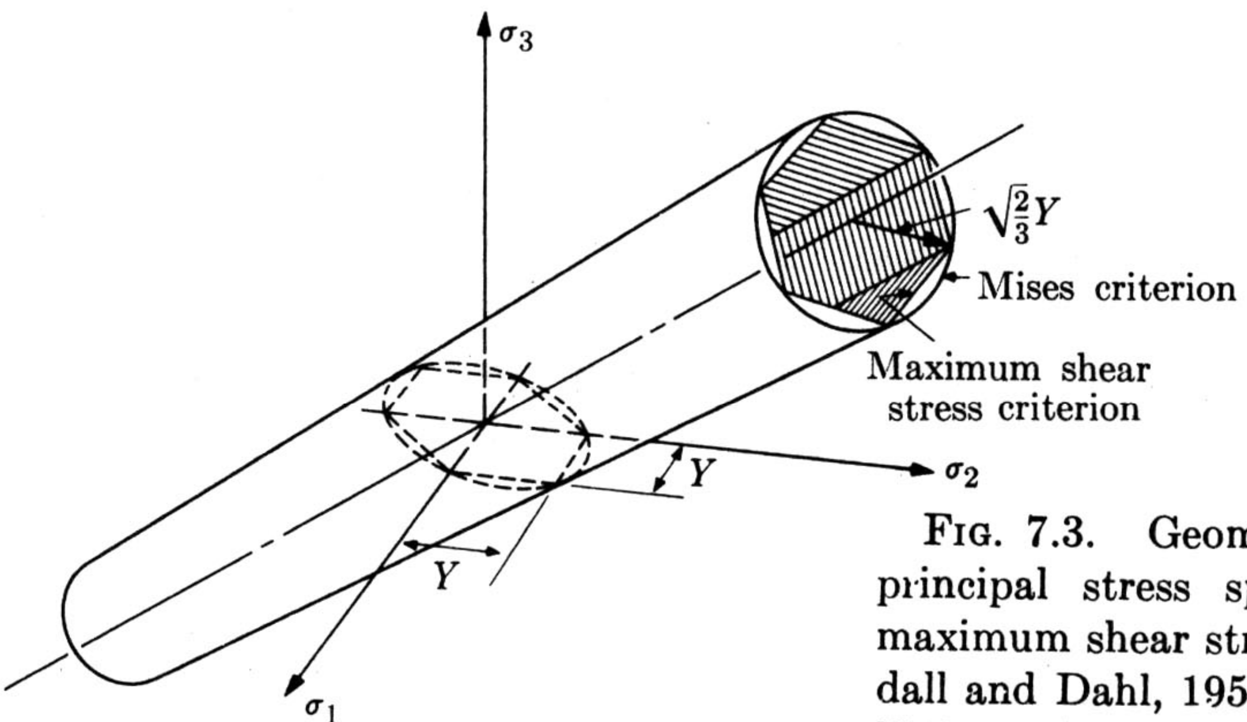
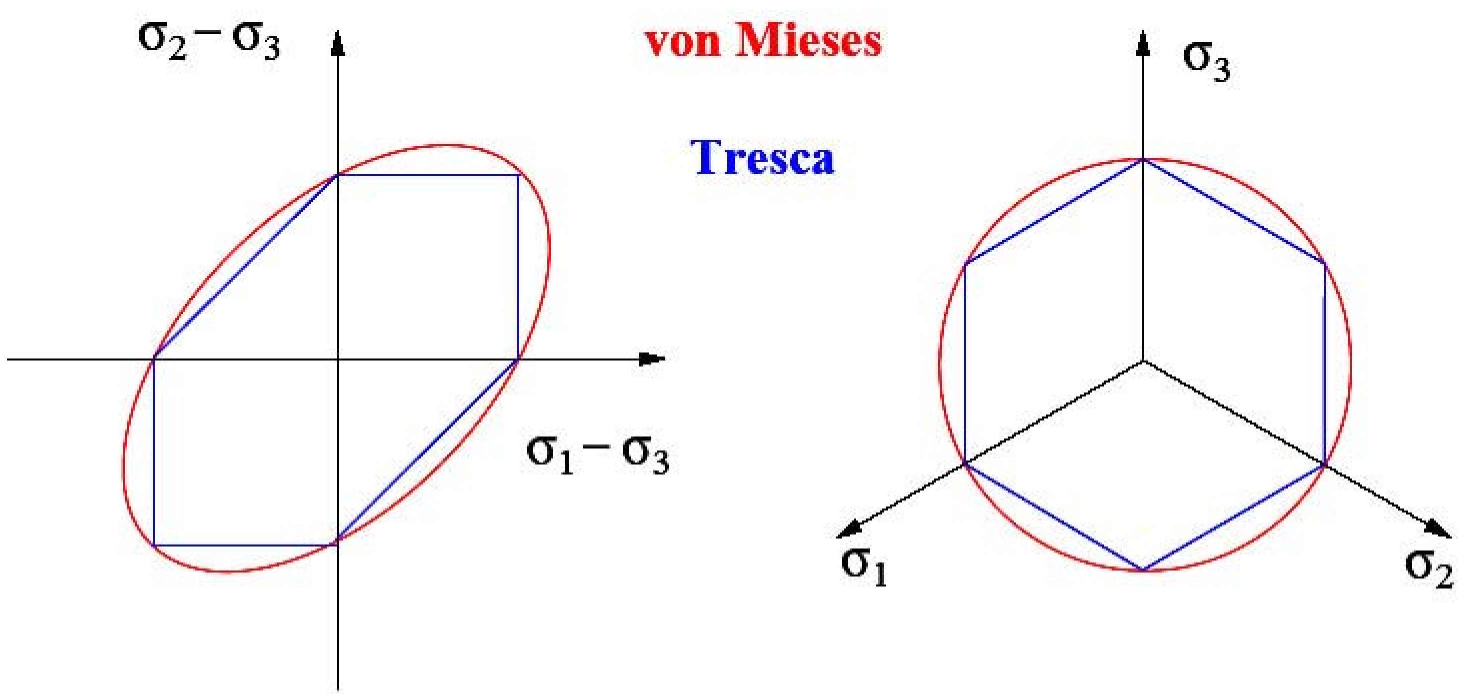


FIG. 7.3. Geometrical representation in principal stress space of the Mises and maximum shear stress yield criteria. (Crandall and Dahl, 1959. Courtesy of McGraw-Hill.)

Constitutive Equations

Physics of plasticity of metals- Yield Criteria



Advantage of Mises criterion: Continuous function

Advantage of Tresca criterion: Simple

Constitutive Equations

Physics of plasticity of metals- Yield Criteria

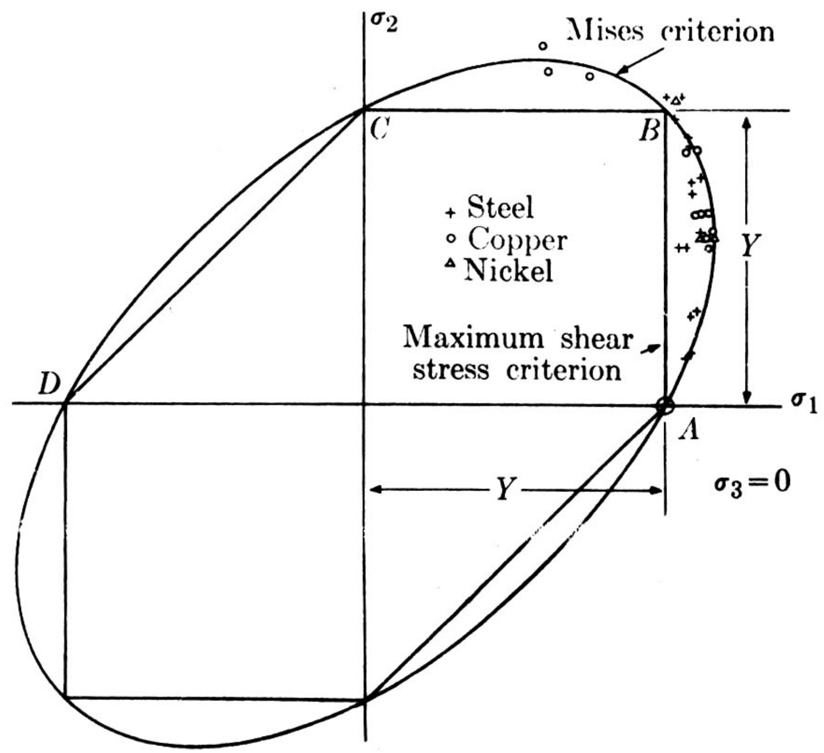


FIG. 7.2. Yielding of thin-walled tubes under combined stresses. (Lode, 1926; from Crandall and Dahl, 1959. Courtesy of McGraw-Hill.)

Mises criterion is more conservative

Constitutive Equations

Physics of plasticity of metals- Tensile yield and shear yield

Tensile yield σ_y

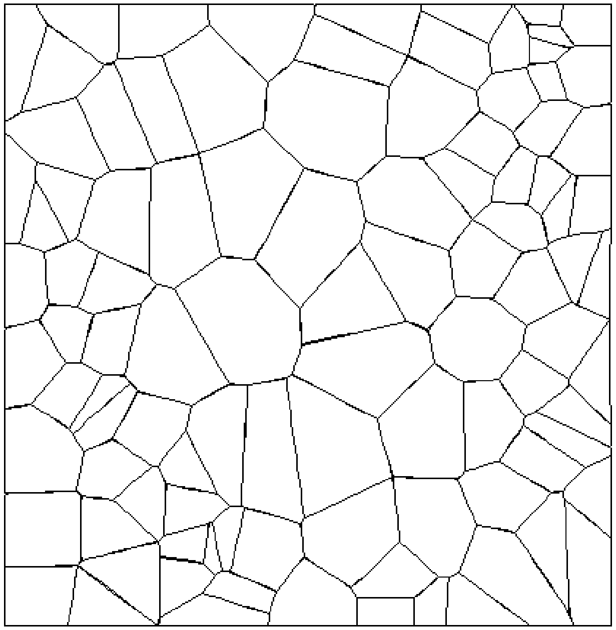
Shear yield τ_y

Constitutive Equations

Physics of plasticity of metals- Single crystal and polycrystal

Shear yield strength of polycrystal $\tau_{y,poly}$

Shear yield strength of single crystal $\tau_{y,\mathrm{single}}$



Constitutive Equations

Plastic Deformation:

- Elastic strain in a single crystal is the strain related to the stretching of the crystal lattice under the action of applied stress. Therefore, elastic strain is recoverable.

- Since the production of plastic strain requires the breakage of interatomic bonds, plastic deformation is dissipative.

- Crystals contain dislocations; When dislocations move, the crystal deform plastically.

- Plastic strain is incompressible because dislocation motion produces a shearing type of deformation; Hydrostatic pressure has a negligibly small effect on the plastic flow of metals.

- Plastic strain remains upon removal of applied stress, that is, plastic strain produces a permanent set.

- The current resolved shear stress governs the plastic strain increment, and not the total strain as in linear elasticity.

Constitutive Equations

Rate-Dependent Plasticity and Rate-Independent Plasticity:

Plastic deformation in metals is thermally-activated and inherently rate-dependent

- Energy barrier and thermal vibration of atoms
- Increasing temperature promote thermal vibration
- Applying stress lowers down the energy barrier

For most single and polycrystalline materials, the plasticity is only slightly rate-sensitive if the temperature

$$T < 0.35\, T_m$$

Constitutive Equations

General Requirements for a Constitutive Model:

Requirements from thermodynamics

First Law: The rate of change to total energy of a thermodynamic system equals the rate at which external mechanical work is done on that system by surface tractions and body forces plus the rate at which thermal work is done by heat flux and hear sources.

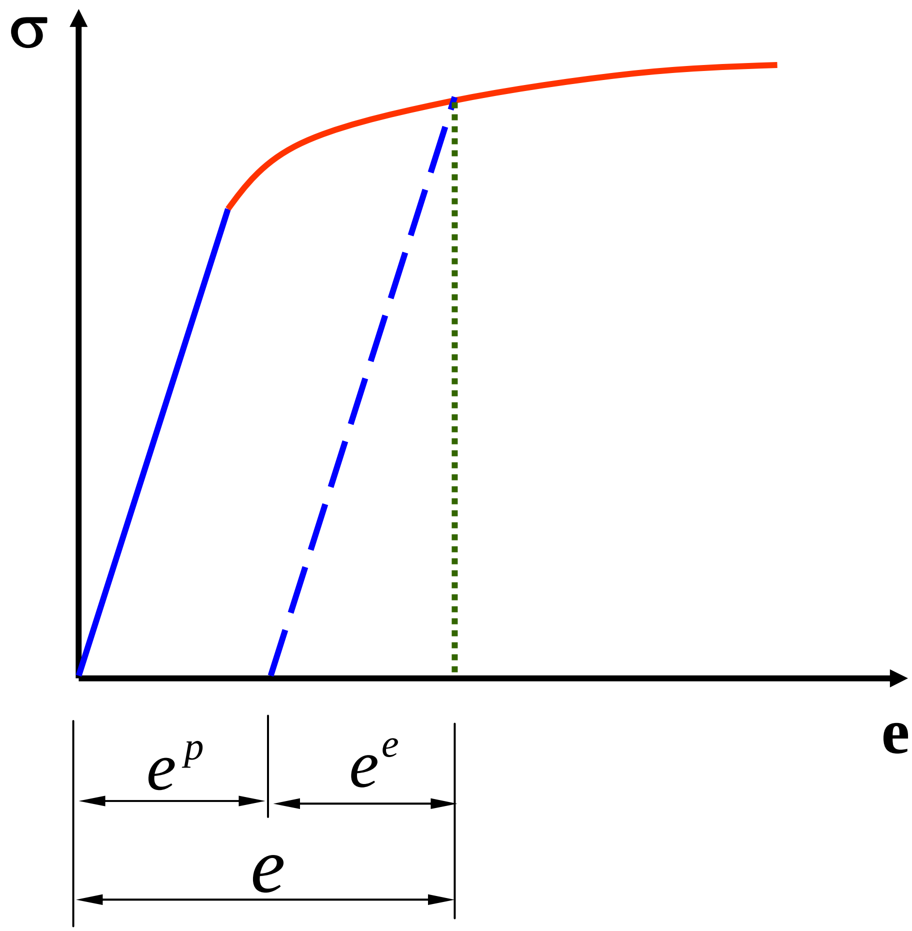
Total energy change = Mechanical work + Heat transfer

Second Law: (Entropy inequality principle).
Entropy cannot decrease unless some work in done;
Heat always flows from the warmer to the colder region of a body, not vice versa;
Mechanical energy can be transformed into heat by friction, and this can never be converted back into mechanical energy.

Energy dissipation >=0

Constitutive Equations

One Dimensional Theory of Isothermal Rate-Independent Plasticity:



Constitutive Equations

One Dimensional Theory of Isothermal Rate-Independent Plasticity:

1. Constitutive Equation For Stress

Stress is due to elastic deformation.

Or

Constitutive Equations

One Dimensional Theory of Isothermal Rate-Independent Plasticity:

2. Yield Condition:

Yield Function: $f\left(\sigma - s\right) = \bar{\sigma} - s$

Yield Condition: $f\left(\sigma - s\right) = \bar{\sigma} - s = 0$

s: Deformation resistance.

Internal variable.
Non-zero, positive valued scalar with dimension of stress

Internal variables:

Internal variables cannot be directly observed. They describe the internal structure of materials associated with irreversible effects.

Constitutive Equations

One Dimensional Theory of Isothermal Rate-Independent Plasticity:

3. Evolution Equation For e^p, Flow Rule:

$$\dot{e}^p = \dot{\bar{e}}^p \operatorname{sign}(\sigma) \qquad \dot{\bar{e}}^p \geq 0$$

Constitutive Equations

One Dimensional Theory of Isothermal Rate-Independent Plasticity:

4. Evolution Equation For s, Hardening Rule:

$$\dot{s} = h\dot{\bar{e}}^{p}$$

h: Hardening function

Constitutive Equations

One Dimensional Theory of Isothermal Rate-Independent Plasticity:

5. Consistency Condition:

Constitutive Equations

One Dimensional Theory of Isothermal Rate-Independent Plasticity:

5. Consistency Condition:

Constitutive Equations

One Dimensional Theory of Isothermal Rate-Independent Plasticity:

Summary of 1-D theory

1. Decomposition of strain $$e = e^p + e^e$$

2. Constitutive Equation $$\dot{\sigma} = E\left(\dot{e} - \dot{e}^p\right)$$

3. Yield Condition: $$f = \bar{\sigma} - s \leq 0$$

4. Flow Rule : $$\dot{e}^p = \dot{\bar{e}}^p \mathrm{sign}\left(\sigma\right) \qquad \dot{\bar{e}}^p \geq 0$$

$$\dot{\bar{e}}^p = \begin{cases} 0 & if \;\; f < 0 \\ 0 & if \;\; f = 0, and, sign\left(\sigma\right)\dot{\sigma}^{trial} < 0 \\ g^{-1} sign\left(\sigma\right)\dot{\sigma}^{trial} & f \;\; f = 0, and, sign\left(\sigma\right)\dot{\sigma}^{trial} > 0 \end{cases}$$

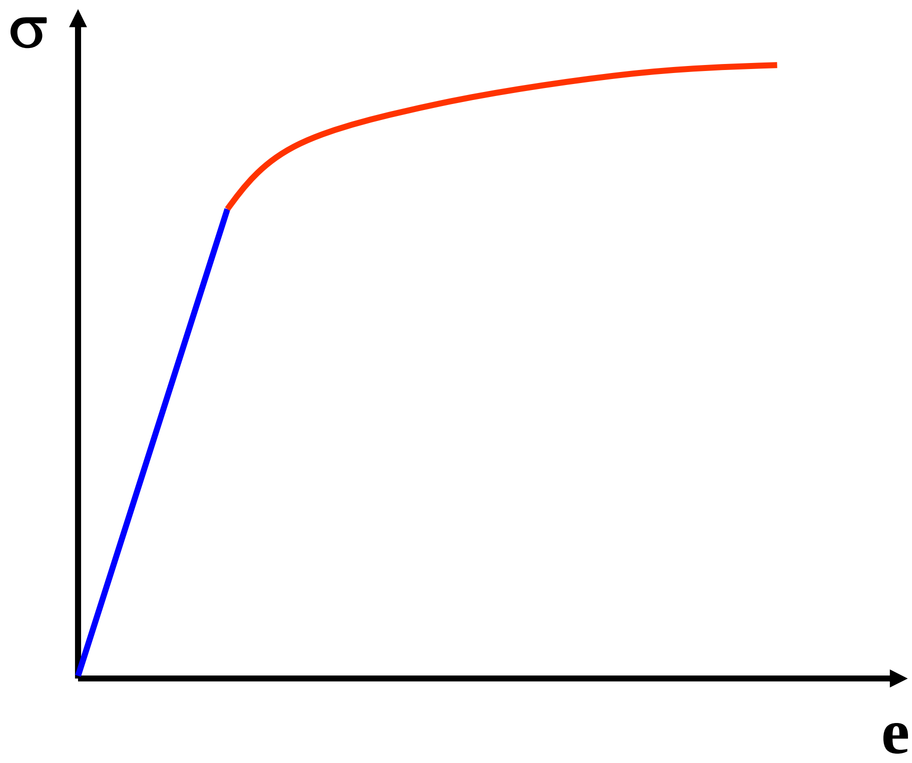
$$\dot{\sigma}^{trial} = E\dot{e} \qquad g = E + h$$

5. Hardening Rule s: $$\dot{s} = h\dot{\bar{e}}^p$$

Constitutive Equations

One Dimensional Theory of Isothermal Rate-Independent Plasticity:

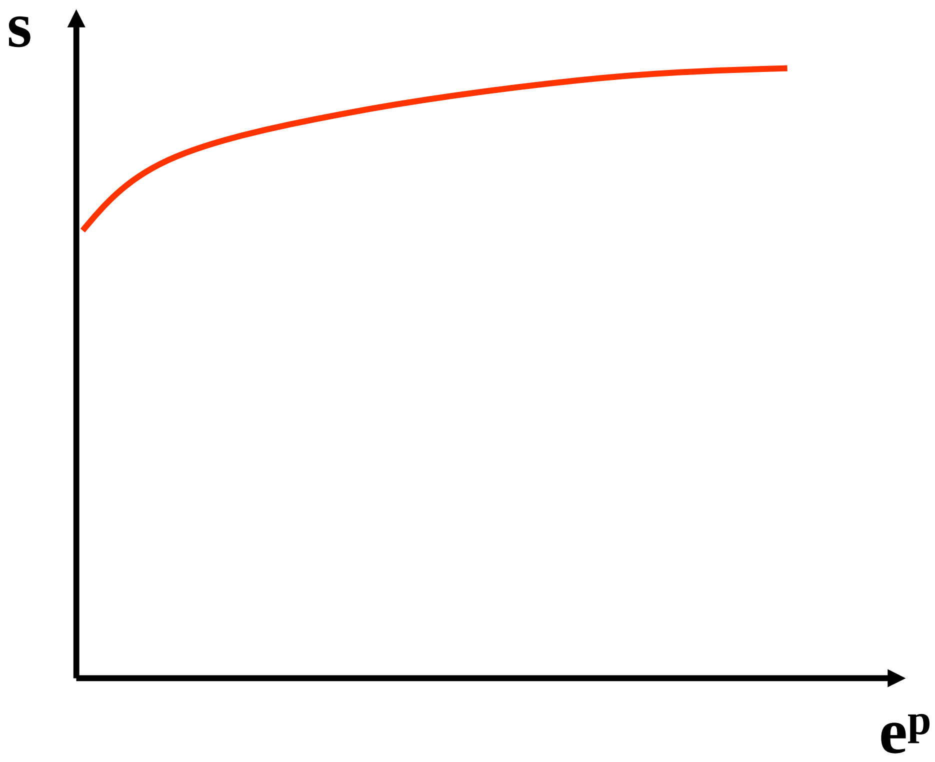
What are the material parameters and how do we determine them?



1. Uniaxial tension or compression

Constitutive Equations

One Dimensional Theory of Isothermal Rate-Independent Plasticity:



$$h = \frac{ds}{de^p}$$

Constitutive Equations

One Dimensional Theory of Isothermal Rate-Independent Plasticity:

Possible functions for *h for metals*

$$h = h_0 \left(1 - \frac{s}{s*} \right)^{\alpha}$$

$$s = s^* - \left[\left(s^* - s_0 \right)^{1-\alpha} + \left(\alpha - 1 \right) h_0 s^* \overline{e}^{\,p} \right]^{\frac{1}{1-\alpha}}$$

Constitutive Equations

One Dimensional Theory of Isothermal Rate-Independent Plasticity:

Another Hardening function

$$s = K\left(\bar{e}^{\,p}\right)^{n}$$

Constitutive Equations

Three Dimensional Theory of Isothermal Rate-Independent Plasticity:

Governing Variables

Constitutive Equations

Three Dimensional Theory of Isothermal Rate-Independent Plasticity:

1. Decomposition of strain rate: $$e = e^p + e^e \qquad \text{(1D)}$$

2. Constitutive Equations $$\dot{\sigma} = E\left(\dot{e} - \dot{e}^p\right) \qquad \text{(1D)}$$

Constitutive Equations

Three Dimensional Theory of Isothermal Rate-Independent Plasticity:

3. Yield Condition: $f = \left|\sigma\right| - s \leq 0$ (1D)

Constitutive Equations

Three Dimensional Theory of Isothermal Rate-Independent Plasticity:

4. Flow Rule :

$$\dot{e}^{p} = \dot{\bar{e}}^{p}\,\mathrm{sign}(\sigma) \qquad \text{(1D)}$$

Constitutive Equations

Three Dimensional Theory of Isothermal Rate-Independent Plasticity:

4. Flow Rule (continued):

Constitutive Equations

Three Dimensional Theory of Isothermal Rate-Independent Plasticity:

5. Hardening Rule s:

$$\dot{s} = h\dot{\bar{e}}^{p}$$

Constitutive Equations

Three Dimensional Theory of Isothermal Rate-Independent Plasticity:

Summary of 3-D theory

1. Decomposition of strain

$$e_{ij} = e_{ij}^{p} + e_{ij}^{e}$$

2. Constitutive Equation

$$\dot{\sigma}_{ij} = 2G\left(\dot{e}_{ij} - \dot{e}_{ij}^{p}\right) + \lambda \dot{e}_{kk}\delta_{ij}$$

3. Yield Condition:

$$f = \left|\sigma\right| - s \leq 0$$

4. Flow Rule :

$$\dot{e}_{ij}^{p} = \dot{\bar{e}}^{p}\text{sign}\left(\frac{3}{2}\frac{\sigma'_{ij}}{\bar{\sigma}}\right) \qquad \dot{\bar{e}}^{p} \geq 0$$

$$\dot{\bar{e}}^{p} = \begin{cases} 0 & if \;\; f < 0 \\ 0 & if \;\; f = 0, and, \sigma'_{ij}\dot{\sigma}_{ij}^{trial} < 0 \\ g^{-1}\dfrac{3}{2}\dfrac{\sigma'_{ij}}{\bar{\sigma}}\dot{\sigma}_{ij}^{trial} & f \;\; f = 0, and, \sigma'_{ij}\dot{\sigma}_{ij}^{trial} > 0 \end{cases}$$

$$\dot{\sigma}_{ij}^{trial} = 2G\dot{e}_{ij} + \lambda\dot{e}_{kk}\delta_{ij} \qquad g = 3G + h$$

5. Hardening Rule s:

$$\dot{s} = h\dot{\bar{e}}^{p}$$