Interactions among topographically induced elastic stress, static fatigue, and valley incision

Peter Molnar

Department of Geological Sciences, Cooperative Institute for Research in Environmental Science, University of Colorado at Boulder, Boulder, Colorado, USA

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[1] Gravity acting on topography creates differential stress, which, for sufficiently high, steep topography, can fracture intact bedrock. If static fatigue governed the time dependence of that fracturing, a feedback should exist between incision of the landscape and fracturing. The timescale for static fatigue decreases exponentially with differential stress. By creating deeper, steeper valleys, incision should increase the differential stress and, as a result, accelerate fracturing of intact rock. Because both fluvial systems and hillslopes can transport fractured rock much more easily than they can both erode intact rock and then transport its products, topographically induced fracturing could play a crucial role in landscape evolution, especially of steep terrain. Predicting differential stress useful for general rules in geomorphology, however, appears to be difficult, in part because the stress distribution depends sensitively on the precise form of the topography and, perhaps more importantly, because the differential stress for specific topography can differ not only in magnitude but also in sign for small differences in Poisson's ratio. Nevertheless, it appears possible that in steep terrain where rivers are capable of moving bed load sufficiently rapidly, the rate-limiting process for incision might be static fatigue of the rock under stress due to gravity acting on the adjacent topography. *INDEX TERMS:* 1815 Hydrology: Erosion and sedimentation; 5104 Physical Properties of Rocks: Fracture and flow; 8010 Structural Geology: Fractures and faults; 1625 Global Change: Geomorphology and weathering (1824, 1886); 8164 Tectonophysics: Stresses—crust and lithosphere; *KEYWORDS:* topography, static fatigue, river incision

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1. Introduction

[2] Steep valleys in mountainous terrain would fill with sediment from neighboring hillslopes were sediment transport not fast enough to remove it. Observations from hillslopes in regions of steep terrain suggest that slopes steeper than $\sim 30° - 45°$ are rare and that debris on such slopes continually moves downward into the valley floors [e.g., *Burbank et al.*, 1996; *Schmidt and Montgomery*, 1995]. Sustaining such steep slopes therefore requires that rivers not only remove that material but also lower valley floors relative to the adjacent rock. Without excavation of bedrock from beneath the valley floor, hillslopes would become gentler, and valleys would fill with debris. Thus the rate of bedrock removal must be a rate-limiting process not only for valley incision, but also for erosion of high-relief terrain in general.

[3] Ron Shreve (personal communication, 1999) pointed out a simple observation that seems to be either poorly appreciated or widely known but ignored by most Earth scientists: Bedrock crops out along only small fractions of most valley floors, and instead, sediment covers the floors of most reaches of valleys. A few studies specifically aimed at bedrock erosion have noted only local exposure of bedrock [e.g., *Wohl*, 1992a, 1992b, 1993; *Wohl et al.*, 1994, 1999], but to my knowledge, there has been no quantitative study of the extent of bedrock exposure. This cover reflects, at least in part, the rarity of events during which flowing water and sediment carried by it can make contact with the bedrock; presumably, only very large floods can move the bed load sufficiently to expose the underlying bedrock, and then, as such floods subside, the bed load falls back onto the bedrock [e.g., *Baker*, 1977; *Baker and Pickup*, 1987; *Hancock and Anderson*, 2002; *Howard*, 1998; *Wohl*, 1992a, 1992b, 1993]. Thus not only are rivers given short spans of time to erode bedrock, but before they can do so they must scour the overlying sediment. R. Shreve recognized that rare floods would be more effective erosive agents if their task were merely to move bed load so that other processes effected the conversion of bedrock to bed load.

[4] In a landmark study, *Miller and Dunne* [1996] showed how topographically induced differential stress [e.g., *Savage et al.*, 1985; *Savage and Swolfs*, 1986] can, at least in some situations, be sufficient to cause fracturing of intact rock, which in turn would be more susceptible to erosion than solid bedrock. If such fracturing converted

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bedrock into bed load, then the burden placed on rivers would be lightened; they would serve as transport, but not erosive, agents. Moreover, as *Miller and Dunne* [1996] recognized, increased incision, for instance, by more rapid bed load transport, could feed back positively with incision; higher differential stress would induce more rapid fracturing of the bedrock and its conversion to bed load.

[5] *Miller and Dunne* [1996] did not discuss the time dependence of failure, which could limit the rate of bed load production. I examine one approach to introducing time dependence, by considering static fatigue. As the number of uncertain quantities that must be assumed prevent the derivation of general relationships for the time dependence of incision that can then be calibrated easily, my goals merely include (1) showing how topographic stress might fracture bedrock over geologically relevant time intervals, using the stress dependence of static fatigue to limit incision rates where bed load transport rates are high, and (2) revealing the uncertainties inherent in quantifying such a process when applying it to real landforms.

2. Topographically Induced Stress

[6] In the equation of equilibrium, which describes the balance between the body force per unit mass and gradients in stress, gravity acting on lateral variations in topography requires that there be gradients in stress to support the topography. Solving the equation of equilibrium for such stress distributions, however, poses a difficult problem because the topography requires the application of boundary conditions on a surface that is not planar. To illustrate some basic principles and sources of uncertainty, let us consider analytic solutions for stress associated with two examples of idealized two-dimensional topography.

[7] Consider first stresses in a half-space with gravity perpendicular to its surface and with a two-dimensional cylindrical valley cut into it (Figure 1). Following an approach of R. C. Fletcher (personal communication, 2002), the equations of equilibrium can be written in polar coordinates:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\vartheta}}{\partial \vartheta} + \frac{\sigma_{rr} - \sigma_{\vartheta\vartheta}}{r} + \rho g \sin\vartheta = 0 \tag{1a}$$

$$\frac{\partial \sigma_{r\vartheta}}{\partial r} + \frac{\sigma_{r\vartheta}}{r} + \frac{1}{r}\frac{\partial \sigma_{\vartheta\vartheta}}{\partial \vartheta} + \rho g \cos\vartheta = 0. \tag{1b}$$

Tension is positive, R is the radius, and therefore depth, of the valley, and ϑ is measured counterclockwise relative to the horizontal (Figure 1). Although, in general, the two equations of equilibrium are insufficient to define uniquely the three components of stress (σ_{rr}, $\sigma_{\vartheta\vartheta}$, and $\sigma_{r\vartheta}$), they do suffice with the assumption that lithostatic equilibrium (negligible shear stress) holds far from the cylinder and with the boundary conditions of vanishing shear and normal tractions on the free surface. We may test whether $\sigma_{r\vartheta} = 0$ everywhere, not just at $r = R$, yields a solution. Rewriting equation (1b) as

$$\frac{\partial \sigma_{\vartheta\vartheta}}{\partial \vartheta} = -\rho g r \cos\vartheta \tag{2}$$

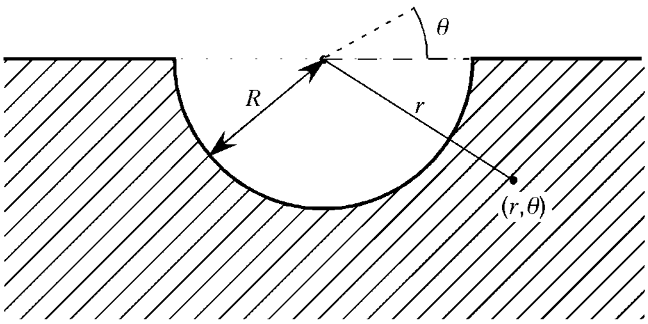

Figure 1. Topographic profile across a cylindrical valley and coordinate system used to describe the state of stress.

with the boundary condition that $\sigma_{\vartheta\vartheta}(r, \vartheta = 0) = 0$, equation (2) yields $\sigma_{\vartheta\vartheta} = -\rho g r \sin\vartheta$. Accordingly, equation (1a) becomes

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{r} = -2\rho g \sin\vartheta. \tag{3}$$

With the boundary condition that $\sigma_{rr}(r = R, \vartheta) = 0$, the complete solution is given by

$$\sigma_{rr} = -\rho g r \sin\vartheta + \rho g \frac{R^2 \sin\vartheta}{r}, \tag{4a}$$

$$\sigma_{\vartheta\vartheta} = -\rho g r \sin\vartheta, \tag{4b}$$

$$\sigma_{r\vartheta} = 0. \tag{4c}$$

Note that for this case, no assumption of a constitutive relationship has been made. Moreover, although the valley is instantaneously inserted into the landscape, the stress distribution given by equation (4) applies regardless of the temporal development of the valley.

[8] This solution illustrates two simple aspects of topographically induced stresses. First, the differential stress ($\sigma_{rr} - \sigma_{\vartheta\vartheta} = \rho g R^2 \sin\vartheta/r$) decreases inversely with distance from the axis of the valley, from its maximum at the walls of the valley $r = R$, where it is proportional to the magnitude of relief, the depth of the valley R. Second, in this case the normal stresses are compressive (negative). As most solids, including rock, are much weaker in tension than in compression, the absence of tension suggests that a valley of this shape should be relatively strong.

[9] As a second example, let us consider the same family of topographic profiles that *Miller and Dunne* [1996] used. *Savage et al.* [1985] derived a solution for the state of stress in an elastic solid with uniform elastic constants in a constant gravity field and with a particular class of two-dimensional topographic profiles that includes both ridges and valleys (Figure 2):

$$x = u + \frac{abu}{u^2 + a^2} \tag{5a}$$

$$z = \frac{a^2 b}{u^2 + a^2}. \tag{5b}$$

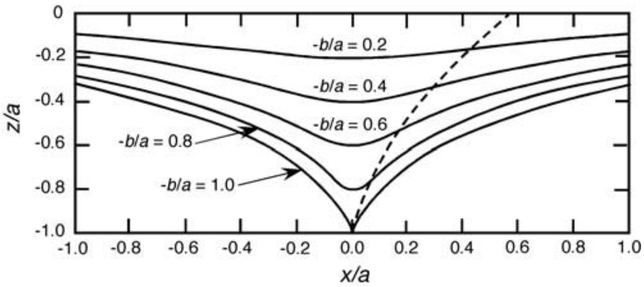

Figure 2. Topographic profiles, scaled by the parameter a, for which *Savage et al.* [1985] obtained a solution for stress, for values of $-b/a$ = 0.2, 0.4, 0.6, 0.8, and 1.0. The dashed line shows the locus of inflection points, where the topographic slopes reach their maximum values.

In a Cartesian coordinate system, x points horizontally perpendicular to the ridge or valley and z points upward, opposite to gravity; a (>0) and b define the shape of the topographic cross section (Figure 2); and u serves as a dummy variable. A ridge corresponds to $b > 0$ and a valley to $-a < b < 0$ such that the height of the ridge or the depth of the valley is equal to $|b|$. Solutions for $b < -a$ are disallowed. Obviously, the ratio $a/|b|$ then determines the steepness of the valley, with large values giving wide, gentle valleys and values approaching 1 giving the steepest valleys that can be considered. For topography defined by equation (5), the topographic slope reaches a point of inflection ($d^2z/dx^2 = 0$) where $u = \sqrt{a(a+b)/3}$ (Figure 2).

[10] To solve for the state of stress, *Savage et al.* [1985] used the complex variable $x + iz$ and transformed the topography to a flat surface in (u, v) coordinates, with v pointing upward. Using the complex variable $w = u + iv$, they obtained the stress distribution, and via conformal mapping, they then transformed the solution back to (x, z) coordinates. The complicated algebra of their solution, given in Appendix A, makes its patterns less transparent than does that for a cylindrical valley, but *Martel and Muller* [2000] reproduced it using the boundary element method and explained some of its possibly counterintuitive aspects. *Savage and Swolfs* [1986] extended the solution of *Savage et al.* [1985] to include a regional tectonic stress, but I do not examine that effect.

[11] One may understand the stress distribution that *Savage et al.* [1985] obtained as consisting of two parts: an initial, or background, state of the stress in a half-space with a horizontal surface plus that associated with removal of material to leave topography [*Martel and Muller*, 2000]. For an initial state they assumed that

$$\sigma_{zz} = \rho g z \tag{6a}$$

$$\sigma_{xx} = \sigma_{yy} = \rho g z \frac{\nu}{1-\nu}, \tag{6b}$$

where ν is Poisson's ratio. Instead, I consider an initial state of lithostatic equilibrium, and hence with no differential stress,

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \rho g z. \tag{7}$$

For $\nu \neq 0.5$ the difference between equations (6) and (7) obviously can be large at great depths. In any case, as for the cylindrical valley discussed above, the state of stress cannot be uniquely determined from knowledge of topography alone [e.g., *Jeffreys*, 1962, p. 196], and the stresses that I consider here should be seen as changes in stress due to excavation of a valley by erosion. That excavation need not be instantaneous, but I employ the static stress after the valley has been created, and hence I ignore any preceding time development. In addition, the explicit consideration of stress within intact rock, which contains small, isolated, and nearly closed cracks but not joints and large interconnected fractures, denies such fractures a major role. Accordingly, I ignore the role of pore pressure [e.g., *Hubbert and Rubey*, 1959] and its contribution to shear failure in valleys where rock is highly fractured and pore pressure reduces the effective shear stress [e.g., *Iverson and Reid*, 1992; *Reid and Iverson*, 1992]. I presume that such fractures result from failure of intact rock induced by topography and that only then does fluid pressure affect failure of valley walls.

[12] As with the cylindrical valley, all perturbations to the stress distribution due to the topography scale with the magnitude of topography and therefore for the valley represented by equation (5) with $\rho g|b|$ (see Appendix A).

[13] Given the complexity of the algebraic expression for the stress distribution, to gain insight, let us consider the state of stress directly beneath a valley floor, at $x = 0$, where, by symmetry, σ_{xz} (0, z) = 0. Let us first consider the stresses at the Earth's surface at the bottom of a valley where $z = b < 0$, in equation (5b). The boundary condition of no normal traction requires that σ_{zz} (0, b) = 0, and the horizontal stress is obtained by evaluating equations (A2) and (A3) at $w = 0$:

$$\frac{\sigma_{xx}(0,b)}{\rho g b} = \frac{b^2 - 2b^2\nu + 2ab - 5ab\nu}{(1-\nu)(a+b)(2a+b)}. \tag{8}$$

Although perhaps not enlightening without inserting numbers, for typical values of Poisson's ratio for intact rock (0.25–0.3) [*Turcotte and Schubert*, 1982, p. 432], $\sigma_{xx}(0, b)$ is positive and hence tensile at the floor of a valley. Given the much lower tensile than compressive strength of rock, this fact shows that relatively simple and smooth topography can facilitate fracture at the Earth's surface. Moreover, for narrow valleys, as $a \rightarrow -b$, the horizontal normal stress σ_{xx} (0, b) becomes much larger than that for wide valleys of the same depth (Figure 3). As *Miller and Dunne* [1996] noted, gravity acting on topography can cause large stresses, and even for smoothly varying topography, such stresses can exceed the strength of intact rock.

[14] Equation (8) contains a strong dependence of σ_{xx} (0, b)/$\rho g b$ on Poisson's ratio, ν (Figure 3). First, when ν = 1/3, equation (8) becomes

$$\frac{\sigma_{xx}(0,b)}{\rho g b} = \frac{b}{2(2a+b)} \tag{9}$$

so that the singularity as $a \rightarrow -b$, which seems so impressive for smaller values of ν (Figure 3), vanishes. Then, for $\nu > 1/3$, again there is a stress concentration, but that stress becomes increasingly compressive, not tensile, as $a \rightarrow -b$ (Figure 3). (Recall that no solution exists for $b \leq -a$.)

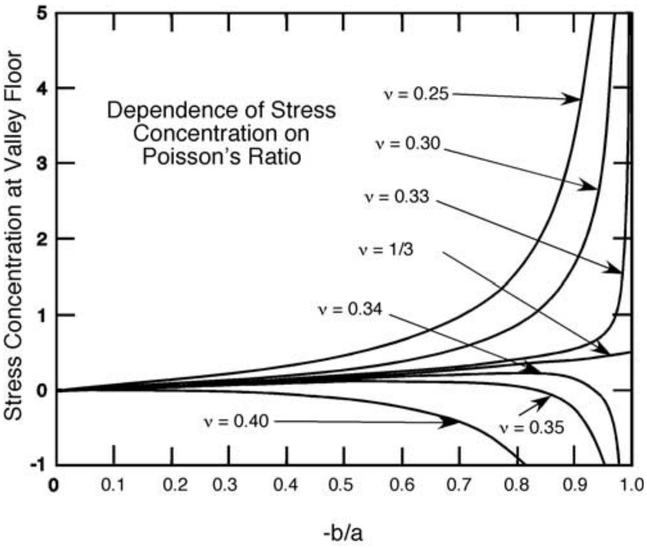

Figure 3. Plot of the stress concentration, the normalized horizontal tensile stress σ_{xx} (0, b)/$\rho g|b|$ given by the right side of equation (8), at the bottom of valleys as a function of $-b/a$ for different values of Poisson's ratio, ν. As valleys narrow ($a \rightarrow -b$), the stress concentration increases at the valley bottom, but the nature of that stress depends on ν such that for $\nu > 1/3$, σ_{xx} (0, b) is tensile, but for $\nu > 1/3$, it is compressive.

[15] The dependence of σ_{xx} (0, b) on ν will affect attempts to estimate elastic stresses from arbitrary topography because open cracks within the rock can affect its value. Where differential stress is low, the presence of open cracks reduces the effective value of Young's modulus (the ratio of axial stress to axial strain) [*Walsh*, 1965a]. Thus the response of the rock to uniaxial stress includes both greater axial compressional (or extensional) strain and less extensional (or compressional) strain with the orthogonal orientations. For tensile stress, Poisson's ratio for rock with open cracks is reduced below that of intact rock [*Walsh*, 1965b]. For large compressive stress, however, as cracks both close and undergo shear of one side past the other, Poisson's ratio increases with increasing differential stress [*Walsh*, 1965b]; the slip of one side of the crack with respect to the other transfers axial compression to greater lateral extension. The presence of incompressible fluids within the cracks can enhance this increase in Poisson's ratio. In fact, a common assumption is that because of slow creep of rock under differential stress, Poisson's ratio approaches 0.5 ("Heim's rule" [*Jaeger and Cook*, 1976, p. 368]). Because the calculated stress at the bottom of the valley is tensile, however, the rock should be in *Walsh*'s [1965b] low-stress regime, and if erosion occurs faster than the rock can creep, we should expect $\nu < 1/3$.

[16] As one might intuit, the vertical gradient of the horizontal compressive stress beneath the valley $d\sigma_{xx}$ (0, z)/dz, obtained by differentiating equations (A2) and (A3) by z, also increases as $a \rightarrow -b$ (Figure 4). Although topographically induced tensile stresses can occur at the valley floor, they cannot persist to great depth (Figure 5), and the depth range over which tensile stresses prevail depends strongly on the ratio b/a, with that depth range decreasing as $a \rightarrow -b$ (Figure 6). Thus as a approaches $-b$, horizontal tensile stresses concentrate at the valley bottom in two ways: magnitudes of such stresses increase and so do gradients. Gentle, wide valleys should differ from steep, narrow ones in that higher stresses would be available to fracture the rock at the floors of steep valleys, but tensile stresses can prevail over a greater depth beneath gentler valleys. Whether gentle or steep-sided valleys would be more susceptible to stress-induced fracture of bedrock should depend on the time dependence of failure.

[17] Let us consider magnitudes of stress typical for alpine relief. As noted above, all magnitudes of stresses scale with $\rho g|b|$. Thus for $\rho = 2.7 \times 10^3$ kg m^{-3}, $g = 9.8$ m s^2, and $|b| = 1$ km, tensile stresses and differential stress near the bottoms of valleys will be of order 25 MPa. For $\nu = 0.25$, a typical value, maximum stresses can be a few times larger at the floors of relatively steep valleys (as $a \rightarrow -b$; Figures 5 and 6). For tensile stresses and differential stress to reach 100 MPa, either $-b > 0.9a$ or $|b| > 3$ km (for $0.9a > -b > 0.8a$). Where tensile stresses are large, gradients are also large so that depths over which tension occurs must be shallow (Figure 6). As emphasized above, predicting the stress will be hard where topography is more complex, where the value of ν is uncertain and perhaps spatially variable, and where an additional unknown tectonic stress might be present.

3. Static Fatigue as a Process Regulating the Time Dependence of Failure

[18] Solids subject to sufficiently large differential stress can fail spontaneously by "static fatigue." Experiments

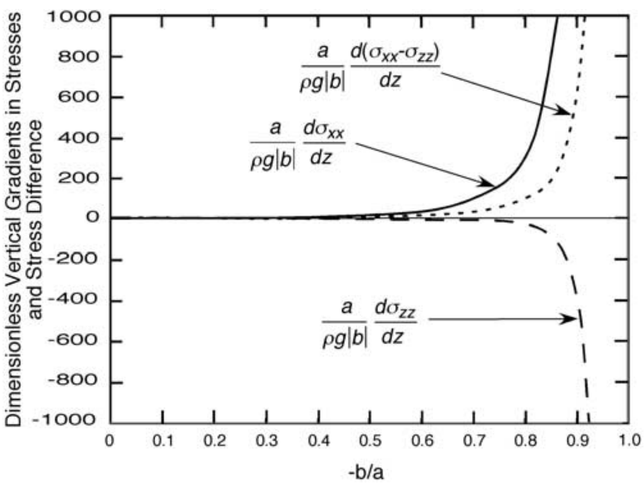

Figure 4. Plots of normalized vertical gradients in stresses at the valley bottom where $z = b$ as a function of $-b/a$ for $\nu = 0.25$, obtained by differentiating equations (A2) and (A3) by z. A positive value of a gradient implies that the component of stress decreases, and hence becomes less tensile, with depth (which increases in the negative z direction). Thus at the valley bottom, where σ_{zz} (0, b) = 0, the negative gradient of this component of stress indicates that it becomes increasingly positive (tensile) with depth. Gradients in stress, even when normalized by $a/\rho g|b|$, increase dramatically as $a \rightarrow -b$.

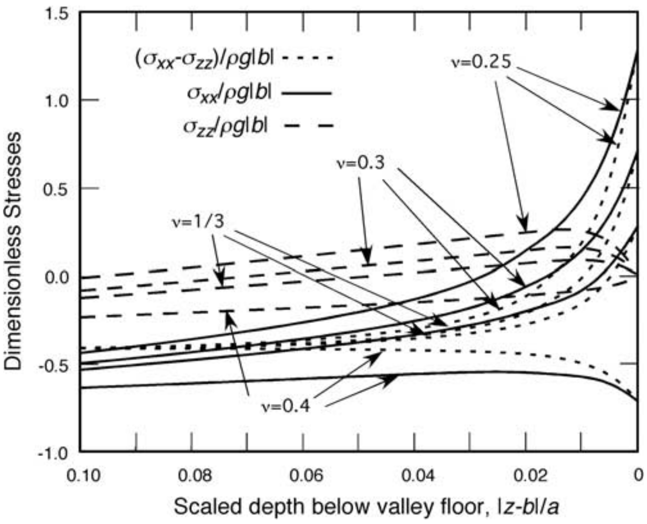

Figure 5. Plots of components for stress and their difference, from equations (A2) and (A3) and all rendered dimensionless using ρgb, beneath a valley as a function of normalized depth beneath the valley for $-b/a = 0.8$ and for four different values of Poisson's ratio, ν. For $\nu \leq 1/3$ the normalized horizontal normal stress, σ_{xx} $(0,\ b)/\rho g|b|$, is tensile at the valley floor and decreases with depth (increasingly negative z), but for $\nu = 0.4$, it is compressive, and its magnitude decreases with depths until the lithostatic load overwhelms the perturbation due to topography. Vertical normal stresses are zero at the valley floor, $\sigma_{zz}(0, b) = 0$. All normal stresses become negative at sufficient depth because lithostatic pressure makes them compressive.

carried out with ceramics and metals, single crystals of rock-forming minerals, and rocks themselves demonstrate that solid materials under sufficient stress deform over time, as cracks within them grow longer [e.g., *Atkinson*, 1984; *Atkinson and Meredith*, 1987; *Costin*, 1987; *Dunning et al.*, 1984; *Freiman*, 1984; *Kranz*, 1980; *Scholz*, 1972, 1990; *Swanson*, 1984]. For high enough stress, cracks propagate at speeds comparable to those for elastic waves, but at lower stress, rates of crack propagation can be slow enough that their dependence on stress and other conditions can be measured in the laboratory. For sufficiently low stress, or more precisely, for sufficiently low stress intensity factors at crack tips, cracks ought not propagate, and the material should not deform over time, though apparently experiments have not yet reached a lower limit for crack growth [*Atkinson and Meredith*, 1987].

[19] Cracks in rocks and rock-forming minerals under stress extend largely because water interacts with the solid material and weakens the bonds. Water can react chemically with silica so that some bonds between silicon and oxygen atoms become bonds between silicon and OH- ions, which are significantly weaker than the Si-O bonds [e.g., *Atkinson*, 1984; *Atkinson and Meredith*, 1987; *Dunning et al.*, 1984; *Freiman*, 1984; *Scholz*, 1990]. Thus the rate at which cracks grow depends on the degree of saturation. (Note that this process is completely different from the effect of fluid pressure in reducing the effective stress [e.g., *Hubbert and Rubey*, 1959].) Both theory and experiment imply that crack growth also depends on temperature because the weakening process depends on the amplitude of vibrations of the atoms in the material at the crack tips.

[20] As *Costin* [1987] summarized, static fatigue under tension is simpler than that for compression. In tension the solid breaks as the cracks forming the weakest links connect. Although tensile cracks can form in solids under compression, particularly where minerals with different elastic constants abut one another [e.g., *Tapponnier and Brace*, 1976], compressive stress tends to close cracks and can retard their growth. As a result, static fatigue under tension is better understood than that under compression. Nevertheless, theories for static fatigue under tension seem to apply to experimental results in uniaxial compression, presumably because in such cases, tensile failure occurs at low confining pressure [*Costin*, 1987].

[21] *Scholz* [1972] applied such theory to measurements that he made using single quartz crystals stressed by uniaxial compression and described the expected lifetime $\langle t\rangle$ of a crystal as

$$\langle t\rangle = C' C_{H_2O}^{-a} \exp\left(\frac{Q_a - k_1\Delta\sigma}{RT}\right). \qquad (10)$$

Here C' is an experimentally determined coefficient, C_{H_2O} is the concentration of water in moles per liter, α is an empirically determined material constant ($\sim$0.3 for quartz [*Scholz*, 1972]), Q_a is an activation energy assuming that static fatigue is a thermally activated process (Q_a = 100 ($\pm$10) kJ mol^{-1} for quartz), R is the universal gas

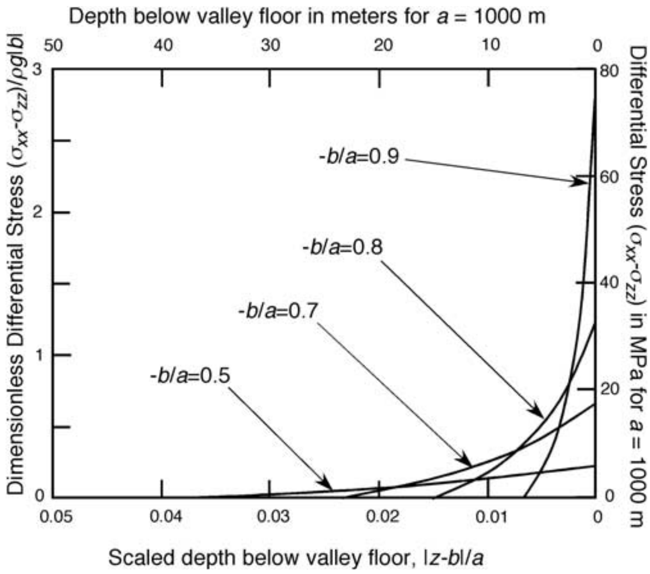

Figure 6. Plots of differential stress, from equations (A2) and (A3), as a function of depth beneath the axis of the valley floor for different values of $-b/a$. The left axis shows normalized values of $(\sigma_{xx}\ (0,\ b) - \sigma_{zz}\ (0,\ b))/\rho g|b|$ versus normalized depths beneath the valley floor on the bottom axis. The right axis shows values of $\sigma_{xx}\ (0,\ b) - \sigma_{zz}\ (0,\ b)$ scaled to a valley width a = 1 km, with depths beneath the valley floor in meters on the top axis and for $\nu = 0.25$. Note that as $a \rightarrow -b$, not only do differential stresses at the valley floor increase and become more tensile, but also the depth range over which they are tensile becomes narrower.

constant, T is absolute temperature, k_1 is an experimentally determined constant, and $\Delta\sigma$ is the differential stress, which in our case for the bottom of a valley would be $|\sigma_{xx} - \sigma_{zz}|$. Strictly, equation (10) applies only to the case where $k_1\Delta\sigma > Q_a$. *Scholz* [1972] focused on the stress dependence of static fatigue and carried out experiments at only two different temperatures, which differ by only 10%, and two different water concentrations. Thus his experiments lent themselves to an expression of the form

$$\langle t \rangle = C' C_{\mathrm{H_2O}}^{-\mathrm{a}} \exp\left(\frac{Q_a}{RT} - k_2\Delta\sigma\right). \tag{11}$$

From measured values of $\langle t \rangle$ for different applied differential stresses, $\Delta\sigma$, *Scholz* [1972] determined that $k_2 = 1.2\ (\pm 0.1) \times 10^{-8}\ \mathrm{Pa}^{-1}$. To be consistent with equation (10), equation (11) applies only when $\Delta\sigma > Q_a/k_2RT$, which, for the values of the parameters *Scholz* [1972] measured and for a typical temperature near the surface of the Earth, requires $\Delta\sigma$ > 3.5 GPa. Such large differential stresses surely are rare within the Earth.

[22] Rock fails at much lower differential stress than do single crystals, by at least one and perhaps by two orders of magnitude [e.g., *Scholz*, 1990, pp. 25–29]. Moreover, quartz is an especially strong mineral. *Jaeger and Cook* [1976, p. 520] reported that the failure of quartz-rich rock depended strongly on the quartz fraction, with pure quartz being an order of magnitude less susceptible to fracture than rock with less than ~30% quartz.

[23] *Kranz* [1980] carried out experiments of static fatigue of granite at atmospheric temperature and humidity and at different confining pressures. He did not examine dependences on either temperature or water saturation. He fit his data using a form similar to equations (10) and (11), but yet simpler:

$$\langle t(\Delta\sigma) \rangle = \langle t \rangle_0 \exp(-k\Delta\sigma). \tag{12}$$

This form ignores a minimum differential stress implied by equations (10) and (11), and thus $\langle t \rangle_0$ serves as maximum lifetime of his samples. Most of the values of k that *Kranz* [1980] obtained were ~10 times smaller than *Scholz*'s [1972] values of k_2 for quartz. Consistent with static fatigue occurring more slowly under triaxial compression than uniaxial compression, he obtained significantly larger values of $\langle t \rangle_0$ than *Scholz*'s [1972] data for quartz imply. *Kranz* [1980] also analyzed data obtained by *Wawersik* [1972] on Westerly granite saturated with water at room temperature and room pressure, and he obtained values of $k = 1.15 \times 10^{-7}\ \mathrm{Pa}^{-1}$ and $\langle t \rangle_0 = 6.3 \times 10^{14}$ s = 20 Myr. I use these values below because the outer crust below river valleys is saturated, and near the surface, confining pressure is negligible.

[24] Let us examine how static fatigue parameterized by equation (12) might affect fracture in an eroding valley. As erosion or incision occurs and rock is removed, the underlying bedrock becomes subjected to an evolving state of stress. Expressions like equations (10)–(12) give expected lifetimes for a solid subjected instantaneously to a constant differential stress, not to an evolving state of stress. Thus we must transform equation (12) into an expression for a time-varying differential stress.

[25] *Scholz* [1972] showed that his data fit an exponential probability distribution:

$$P(t) = 1 - \exp\left(-\frac{t}{\langle t(\Delta\sigma) \rangle}\right). \tag{13}$$

Here $P(t)$ is the probability that a quartz crystal subjected to a differential stress of $\Delta\sigma$ would fail after a time t. Such a probability distribution implies that one must wait an infinite time for all crystals to fail. In the spirit of the simple analysis here, let us assume a simpler form, a uniform probability density function, $P_d(t)$, for failure given by

$$\begin{aligned} P_d(t)dt &= \frac{dt}{\langle t(\Delta\sigma) \rangle}, t \le \langle t(\Delta\sigma) \rangle \\ P_d(t) &= 0, t \ge \langle t(\Delta\sigma) \rangle. \end{aligned} \tag{14}$$

This presumes an equal probability of failure over a finite interval of time, during which all crystals would fail. With equation (13), only 63% of such crystals would fail in the period given by $\langle t(\Delta\sigma) \rangle$. Thus equation (14) calls for somewhat more rapid failure than equation (13). With equation (14), after an elapsed time $t < \langle t(\Delta\sigma) \rangle$ the probability of failure is

$$P(t) = \int_0^t P_d(\tau)d\tau. \tag{15}$$

[26] Now, suppose that $\Delta\sigma$ varies over time. As the differential stress changes, the lifetime of the rock should change, and as $\Delta\sigma$ increases, that lifetime should shorten. Let $t_l(\Delta\sigma(t))$ be the expected lifetime of a sample if a constant differential stress $\Delta\sigma$ were applied at a time t. For the particular history of stress change, we may define the expected lifetime of the rock by the value of time at which the integrated probability density reaches 1:

$$\int_0^{\langle t \rangle} P_d(\tau)d\tau = \int_0^{\langle t \rangle} \frac{d\tau}{t_l(\Delta\sigma(\tau))} = 1. \tag{16}$$

Thus equation (16) yields an implicit equation for $\langle t \rangle$ given the particular history of stress change with time.

[27] To illustrate a simple aspect of equation (16), consider a linearly increasing differential stress over time, $d\Delta\sigma/dt$ = constant:

$$\Delta\sigma(t) = \frac{d\Delta\sigma}{dt}t. \tag{17}$$

Erosion at the bottom of a valley would enhance the stress concentration there. To a first approximation for a constant incision rate, the differential stress would also increase at an approximately constant rate (Figures 5 and 6). Inserting equation (17) into equation (12) and then equation (12) into equation (16) yields an expected lifetime

$$\langle t \rangle = \frac{1}{k\frac{d\Delta\sigma}{dt}} \ln\left(k\frac{d\Delta\sigma}{dt}\langle t \rangle_0 + 1\right). \tag{18}$$

For given material constants k and $\langle t \rangle_0$, when differential stress changes slowly (small $d\Delta\sigma/dt$, so that $k(d\Delta\sigma/dt)\langle t \rangle_0 \ll 1$), $\langle t \rangle$ deviates only slightly from $\langle t \rangle_0$; the effect of relatively high stress at the end of the period to shorten the lifetime is suppressed. When the differential stress changes

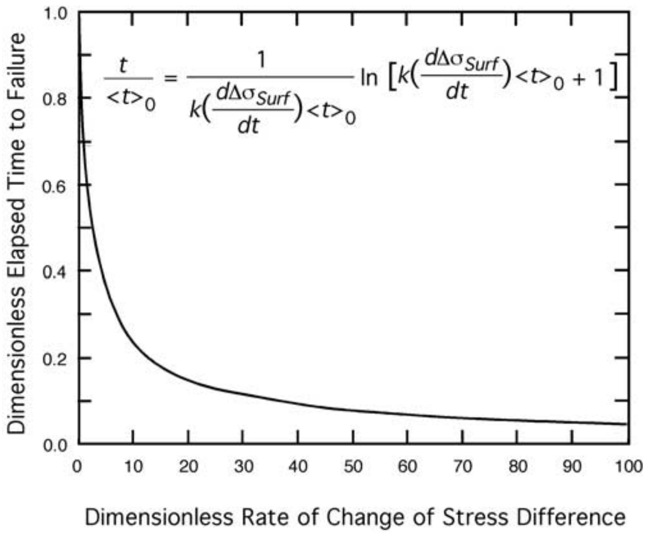

Figure 7. Plot of the dimensionless expected time for failure by static fatigue ($\langle t\rangle/\langle t\rangle_0$) as a function of a dimensionless linear rate of change of differential stress. With $d\Delta\sigma/dt = \Delta\sigma_{surf}/\langle t\rangle$, where $\Delta\sigma_{surf}$ is the differential stress at the Earth's surface and therefore the maximum differential stress, the scaling of $d\Delta\sigma/dt$ uses $k\langle t\rangle_0$. The time to failure decreases rapidly as the rate of stress change increases.

rapidly so that $k\langle t\rangle_0 d\Delta\sigma/dt \gg 1$, $\langle t\rangle \ll \langle t\rangle_0$, and the rapid application of high stress shortens the lifetime of the rock (Figure 7).

4. Interaction of Incision and Failure

[28] We may approximate parameters in equation (18) to estimate the conditions under which failure might lead to positive feedback. Suppose that a differential stress capable of fracturing the rock is confined to a depth range H between the valley floor at $z = b(<0)$ and $z = b - H$, where that differential stress vanishes. To maintain simplicity, let us assume that the differential stress increases linearly over that depth range to the surface, where it reaches the value $\Delta\sigma_{Surf}$ (though clearly from Figures 5 and 6 the gradient is not linear). Then, if the incision rate, E, exceeded the ratio $H/\langle t\rangle$, fresh, unfractured bedrock would be exposed at the valley floor. Conversely, for a slower incision rate the rock exposed at the riverbed should be fractured, and the river's task might merely be to transport the fractured bedrock, which for the purposes of this paper I treat as bed load. Thus for a constant incision rate, where incision consists merely of transporting bed load but not eroding the bedrock, we may assume a limiting condition that

$$\frac{d\Delta\sigma}{dt} \leq \frac{\Delta\sigma_{Surf}}{H}E. \qquad (19)$$

Inserting equation (19) into equation (18), with $\langle t\rangle \leq H/E$, yields

$$\frac{H}{E} \geq \frac{1}{k\frac{\Delta\sigma_{Surf}}{H}E}\ln\left(k\frac{\Delta\sigma_{Surf}}{H}E\langle t\rangle_0+1\right). \qquad (20)$$

Thus an upper bound on the incision rate,

$$E \leq \frac{H}{\langle t\rangle_0}\frac{1}{k\Delta\sigma_{Surf}}\left(e^{k\Delta\sigma_{Surf}} - 1\right), \qquad (21)$$

increases as $\Delta\sigma_{Surf}$ increases (Figure 8).

[29] Using *Kranz*'s [1980] analysis of *Wawersik*'s [1972] data, which suggests that $\langle t\rangle_0 = 20$ Myr, let us assume that $H = 20$ m. Inserted in equation (21), these values of $\langle t\rangle_0$ and H imply a smallest upper bound on $E = 0.001$ mm yr^{-1}. To increase E by 100 times to 0.1 mm yr^{-1}, $k\Delta\sigma_{Surf} \approx 6.47$ so that $\Delta\sigma_{Surf} \approx 56$ MPa, a differential stress that is quite possible in steep mountainous valleys with relief of ~1 km (Figure 6). Consider an incision rate of 1 mm yr^{-1}, which is high compared with a global average erosion rate of ~0.08 mm yr^{-1} (obtained from an accumulated rate of terrigenous sediment accumulation in the ocean of *Hay et al.* [1988] divided by the total area of continents). From Figure 8, $k\Delta\sigma_{Surf} \approx 9.12$, and hence for saturated granite, $\Delta\sigma_{Surf} \approx 79$ MPa, also a plausible value for deep, steep valleys (Figure 6). Finally, for a rapid incision rate of 10 mm yr^{-1}, $k\Delta\sigma_{Surf} \approx 11.67$, corresponding to $\Delta\sigma_{Surf} \approx 101$ MPa; such a stress difference is also plausible, especially in those rare, steep, and deep valleys where such rapid incision occurs, such as in the Nanga Parbat region of

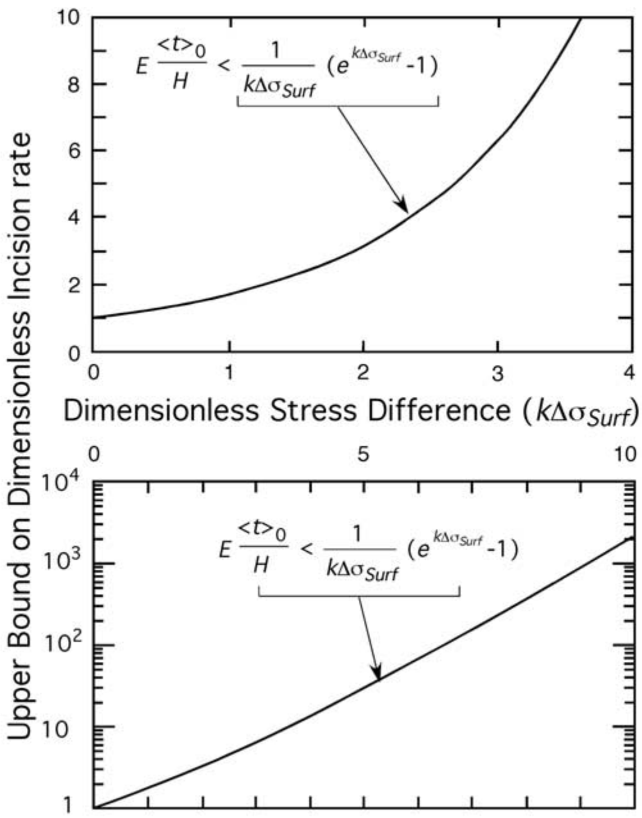

Figure 8. Calculated upper bounds on the incision rate E, normalized by $H/\langle t\rangle_0$, as a function of $\Delta\sigma_{surf}$ and nondimensionalized by k. (top) The rapid increase in the upper bound on E as the maximum differential stress at the surface increases. (bottom) By virtue of the exponential dependence of $\langle t\rangle$ on differential stress in equation (12), at high stress that increase is essentially exponential.

Pakistan [*Burbank et al.*, 1996]. Thus if *Kranz*'s [1980] analysis of *Wawersik*'s [1972] measurements of time-dependent failure of saturated granite applies to field conditions, even moderate erosion rates require large relief. Hence static fatigue under a topographically induced differential stress could be a rate-limiting process of incision where relief is high, slopes are steep, and bed load transport is rapid.

[30] This discussion has ignored uncertainties in the parameters inserted into equation (21). *Kranz*'s [1980] analysis of experiments carried out at different confining pressures yielded values of k that differed by at most a factor of two, which would make the magnitudes of stress uncertain by tens of percent to as much as two times smaller, but not orders of magnitude different, for granite at least. The effect of increasing confining pressure lengthened $\langle t \rangle_0$ by orders of magnitude. Because we are concerned with failure at shallow depths, however, confining pressure should be negligibly small. Of course, the largest uncertainties derive from the assumption that static fatigue is important.

[31] In terms of topographically induced fracturing of rock in eroding valleys, what seems most important is the strong dependence of static fatigue on stress. An increase of only ~20 MPa in stress can shorten the lifetime of unfractured rock by nearly an order of magnitude. Thus as *Miller and Dunne* [1996] recognized, deepening of a valley, which concentrates stress at its bottom, can provide a mechanism for positive feedback. Such changes in stress can arise if a valley with a characteristic width of 1000 m deepens by ~100 m (Figure 6). For more relief, less deepening can effect comparable stress changes and hence can accelerate fracture.

5. Summary and Possible Implications

[32] Two main themes pervade this study. The first is the motivation to quantify *Miller and Dunne*'s [1996] suggestion of positive feedbacks among accelerated incision, increased stress concentration, and fracture of bedrock on the floor of the streambed. The second is the recognition of large uncertainty at each step. If the essence of quantitative science is the quantification of uncertainty, I have made little progress, except to show that within our ignorance of appropriate parameters for magnitudes of stress and time dependence of failure, *Miller and Dunne*'s [1996] suggestion that positive feedback among incision, increased differential stress, and fracture of bedrock might be important.

[33] As *Savage et al.* [1985] showed, differential stress due to gravity acting on topography scales with the magnitude of relief; deeper valleys require greater stresses for their support than do shallower ones with the same depth-to-width scale. In addition, however, the state of stress at shallow depths depends strongly on the particular form of the topography; some topographic forms, like a cylindrical valley (Figure 1), can exist with no tensile stress anywhere, but others, like that exploited by *Savage et al.* [1985], lead to large tensile stress at valley floors. As rock is much weaker under tensile than purely compressive stress, failure is much more likely in those rare places where tension prevails. Moreover, although *Savage and Swolfs* [1986] showed that tectonic stress can be easily added to the formulae given by *Savage et al.* [1985], knowing magnitudes of tectonic stress in Earth will require more knowledge than is available today. Thus a wide range of differential stress should characterize the variety of topography on Earth, and it seems unlikely that any general rule can be found for relating magnitudes of stress to simple aspects of topography.

[34] Stress concentrations can be as large as necessary for fracture to occur, but a strong dependence of the stress concentration on Poisson's ratio can transform a precise knowledge of the topography into huge uncertainty in the stress concentration (Figure 3).

[35] Finally, none of the discussion here has addressed the spacing of fractures in rocks and hence the sizes of blocks of rock that should form when bedrock fractures. If fracturing converts bedrock to bed load for streams then to carry away, obviously, the spacing of fractures will influence the rate at which rivers can remove bed load.

[36] For the reasons given in the preceding three paragraphs, it seems premature to try to implement coupled calculations of topographic stresses and of static fatigue in landscape evolution models. Ignorance at so many levels, to say nothing of the basic premise that static fatigue is important, will make such implementation futile, at least with present-day ignorance of the relevant processes and parameters.

[37] From a positive perspective, magnitudes of topographically induced stress and the dependence of static fatigue on differential stress, at least for granite, support *Miller and Dunne*'s [1996] contention that positive feedback should exist between valley incision, increased differential stress, fracture, and accelerated incision. The strong stress dependence of expected lifetimes of intact rock, which decrease by nearly an order of magnitude for an increase in differential stress of ~20 MPa, should help accelerate erosion. As a valley deepens, the differential stress at the valley floor should increase, which should in turn shorten the lifetime of unfractured rock. Fracturing of the bedrock will make it more susceptible to extraction and transport. Incision rates as fast as 1–10 mm yr^{-1} should be confined to steep, deep valleys, where topographic stresses are large enough to fracture rock at a rate compatible with that at which it must be removed. Correspondingly, whether the rate-limiting process for rapid incision in mountain areas is static fatigue, which results from topographically induced differential stress, or bed load transport will depend on the feedback between them. In principle, bed load transport could occur faster than topographic stress can fracture that bedrock, which would make static fatigue the rate-limiting process. Where relief is large (>1000 m), topographically induced differential stresses can be large enough to fracture rock at a rate fast enough that bed load transport might limit incision.

Appendix A: Equations for the Stress Distribution

[38] The following, taken from *Savage et al.* [1985], gives the change in stress within a half-space in a gravity field due to the creation of topography given by equation (5), using the complex variable $w = u + iv$:

$$x + iz = w + \frac{ab}{w - ia}. \quad \text{(A1)}$$

Note that equation (5) follows from equation (A1), where $\nu = 0$:

$$\sigma_{xx} + \sigma_{zz} = -4\ \mathrm{Re}\left\{ \frac{(w-ia)^2}{(w-ia)^2 - ab} \left[A(w) - \frac{ab\Phi(-ia)}{(w-ia)^2} \right] \right\} \quad \text{(A2)}$$

$$\sigma_{xx} - \sigma_{zz} + 2i\sigma_{xz} = \frac{2(w-ia)^2}{(w-ia)^2 - ab} \cdot \left\{ \begin{array}{l} \left[\frac{\bar{w}(\bar{w}+ia)+ab}{\bar{w}+ia} - \frac{w(w+ia)+ab}{w+ia} \right] \Phi'(w) \\ -B(w) - \Phi(w) - \frac{ab[\Phi(w) - \Phi(-ia)]}{(w+ia)^2} \end{array} \right\}. \quad \text{(A3)}$$

The overbar indicates the complex conjugate, and a prime indicates differentiation with respect to w. In equations (A2) and (A3),

$$\Phi(w) = \frac{-(w-ia)^2}{(w-ia)^2 - ab} \left\{ A(w) + \frac{-ab\Phi(-ia)}{(w-ia)^2} \right\}. \quad \text{(A4)}$$

(Note that equation (A4), which is equation (27) of *Savage and Swolfs* [1986], corrects a sign error in equation (35) of *Savage et al.* [1985].) Also, in equations (A2), (A3), and (A4),

$$\Phi(-ia) = -\rho g b \left[\frac{(4a+b)(1-\nu)+b}{8(1-\nu)(2a+b)} \right], \quad \text{(A5)}$$

$$A(w) = \frac{-\rho g b}{8(1-\nu)} \left\{ \frac{i(1-\nu)(4a+b)}{(w-ia)} + ab \left[\frac{1}{(w-ia)^2} - \frac{2ia}{(w-ia)^3} \right] \right\}, \quad \text{(A6)}$$

$$B(w) = \frac{-\rho g b}{8(1-\nu)} \left\{ \frac{i(1-\nu)(4a+b)}{(w-ia)} + ab(1-2\nu) \left[\frac{1}{(w-ia)^2} - \frac{2ia}{(w-ia)^3} \right] \right\}. \quad \text{(A7)}$$

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P. Molnar, Department of Geological Sciences, Cooperative Institute for Research in Environmental Science, University of Colorado at Boulder, Boulder, CO 80309, USA. (molnar@cires.colorado.edu)