Rock Rheology

GEOL 5700 Physics and Chemistry of the Solid Earth

References:

- Turcotte and Schubert, *Geodynamics*, Sections 2.1,-2.4, 2.7, 3.1-3.8, 6.1, 6.2, 6.8, 7.1-7.4.
- Jaeger and Cook, *Fundamentals of Rock Mechanics*, Chapman and Hall, gives a rather more complete overview of elasticity (most of this in Ch. 2, 4).
- Also Sleep and Fujita's *Principals of Geophysics* provides this in tensor notation (Ch. 9)
- Section 1.3 of Ka rato is relevant to the discussion of viscosity.

This will be a quick overview of the pieces we need to sensibly discuss the deformation of the Earth. We start with the easiest piece, elasticity, and move on to more fluid-like rheologies.

Elasticity

The simplest form of elastic behavior is a spring. As you should recall from physics, the force exerted by a spring is proportional to the displacement of the end of the spring:

$$F = k\,x \tag{1}$$

Of course, within the Earth we need to deal with stresses, which is simply the force per unit area, and strain, which is the fractional change in length relative to an unstressed starting length. Stresses are usually denoted by σ (sometimes by τ), strains by ε, and subscripts denote the component. Thus the normal stress on a face normal to the *x* axis (which is therefore directed along the *x* axis as well) is denoted σ_{xx}. Stresses are a force per unit area and so are (in SA units) N/m^2, which is called a Passcal (Pa). Strains are a ratio and therefore are dimensionless. Strains are often most easily thought of in terms of the displacements. If the displacement field **w**(x,y,z) has components w_x, w_y, and w_z, then

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial w_x}{\partial x}, \varepsilon_{yy} = \frac{\partial w_y}{\partial y}, \varepsilon_{zz} = \frac{\partial w_z}{\partial z} \\
\varepsilon_{xy} &= \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right) \\
\varepsilon_{xz} &= \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x}\right) \\
\varepsilon_{yz} &= \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y}\right)
\end{aligned} \tag{2}$$

Note that these are ignoring higher order terms, so these are correct for infinitesimal objects. Also note that the shear strains ε_{xy}, ε_{xz} and ε_{yz} are angular changes as a rock is deformed; exy can be thought of as motion of the x face in the y direction. Note that rotational terms are ignored for most applications. The rotational terms (ω_{xy}, ω_{xz}, ω_{yz}) have a minus sign in (2) where the equivalent shear strains have a plus.

Under these terms, the equivalent expression to (1) for a rock under uniaxial stress is

$$\sigma_{xx} = E\,\varepsilon_{xx} \tag{3}$$

where E is Young's modulus (which is defined as the ratio of stress to strain for a rock under uniaxial compression). Of course, a rock is a three dimensional object, and when you squish it in one direction, you expect that the other directions might change. How much the rock squishes out is determined by Poisson's ratio, which is the ratio of lateral expansion to longitudinal contraction, $-\varepsilon_{yy}/\varepsilon_{xx}$ if we have an isotropic medium under uniaxial compression along the x axis. If a material is incompressible (its volume doesn't change) the Poisson's ratio is 0.5.

A more general version of basic isotropic elasticity has to include shear stresses as well, which are the forces parallel to the face. So σ_{xy} is the shear stress on the face normal to x directed along the y axis. Because we are assuming equilibrium, there can be no net torque on the rocks, and so $\sigma_{xy} = \sigma_{yx}$.

$$\begin{aligned}
E\varepsilon_{xx} &= \sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\\
E\varepsilon_{yy} &= \sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\\
E\varepsilon_{zz} &= \sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\\
E\varepsilon_{xy} &= \left(1+\nu\right)\sigma_{xy}\\
E\varepsilon_{xz} &= \left(1+\nu\right)\sigma_{xz}\\
E\varepsilon_{yz} &= \left(1+\nu\right)\sigma_{yz}
\end{aligned} \tag{4}$$

Note that it is possible for all the coefficients to vary with orientation (including some that are zero in this isotropic formulation). Thus it is possible in a highly anisotropic medium to need 21 coefficients instead of the 2 terms E and Poisson's ratio (ν) to describe the relationship of stress to strain.

Our isotropic case can also be expressed as stresses defined by the strains:

$$\begin{aligned}
\sigma_{xx} &= \left(2G+\lambda\right)\varepsilon_{xx} + \lambda\left(\varepsilon_{yy} + \varepsilon_{zz}\right)\\
\sigma_{yy} &= \left(2G+\lambda\right)\varepsilon_{yy} + \lambda\left(\varepsilon_{xx} + \varepsilon_{zz}\right)\\
\sigma_{zz} &= \left(2G+\lambda\right)\varepsilon_{zz} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\\
\sigma_{xy} &= 2G\varepsilon_{xy}\\
\sigma_{xz} &= 2G\varepsilon_{xz}\\
\sigma_{yz} &= 2G\varepsilon_{yz}
\end{aligned} \tag{5}$$

where λ and G are Lame's parameters, but G is specially known as the shear modulus (or modulus of rigidity). With some juggling it is straightforward to find that

$$\begin{aligned} \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \\ G &= \frac{E}{2(1+\nu)} \end{aligned} \tag{6}$$

A final important parameter is the bulk modulus (or incompressibility). It controls how the volume of the rock changes with pressure and is defined as

$$\begin{aligned} K &= \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}} \\ &= \lambda + \tfrac{2}{3}G \\ &= \frac{E}{3(1-2\nu)} \end{aligned} \tag{7}$$

Aside: One has to be a bit careful in conventions, as earth scientists routinely use positive for compressional stresses, and occasionally the polarity for the shear stresses reverses (engineers routinely use positive for tensile stresses). A related problem is that the shear strain, ε_{xy}, is half the common engineering shear strain, γ_{xy}.

Stresses and strains can be rotated into a frame where the shear stresses and shear strains go to zero. The axes in this case become the principal stress (or strain) axes; the magnitudes of these stresses (or strains) are denoted by subscripts 1 to 3 (e.g., σ_1, σ_2, σ_3). Usually the most compressive stress is σ_1, the least σ_3, but again this convention is occasionally reversed.

As stresses increase, at some point the rock fails. Failure generally depends on the increase in the deviatoric stress, $|\sigma_{1-}\sigma_3|$, and failure occurs at a stress termed the yield stress. (You should know that the deviatoric stress is twice the maximum shear stress). A simple case is Coulomb failure, which is a modified version of simple frictional laws you may have seen in physics (we'll come back to that later in the course). Beyond failure, many things can happen. First and most generally, deformation becomes permanent: remove the stress and the rock fails to return to its original shape. Elastic deformation is always reversible: remove the stress and strains go to zero. But other deformations are not so easily reversed. Material beyond the yield stress can lose a fair amount of strength, but a more common case is one where deformation continues as long as stress is applied. This case is called plastic deformation; a perfectly plastic deformation will have the strain continue to as long as the yield stress is applied. Material that behaves elastically until a yield stress is applied and beyond that behaves plastically is termed an elastic-plastic material.

Seismology. Before we leave elasticity, one quick note. You can take Hooke's Law (eqn. 4) and combine it with the equations for equilibrium (*F=ma*, in essence, but allow *a* to not be 0 as we usually do in tectonophysical calculations) and you will derive the wave equation, which is the basis for seismology. (You can read about this from a rock mechanics perspective in Ch. 13 of the 3rd edition of Jaeger and Cook).

Creep deformation

As you go deeper in the Earth (below about 10 to 80 km or so), deformation beyond elastic behavior is more complex and more widespread. Stresses cease to be correlated with strains and instead become correlated with strain rates. The two most common formulations of such behavior are Newtonian and power-law creep. The first is useful because problems are often directly solvable; the second seems to better reflect most experimental evidence of rock deformation.

Before proceeding, we should note that you can combine these viscous rheologies with elastic ones to make a visco-elastic medium. For many geodynamics problems, elasticity works well enough (e.g., seismology). For others, viscous creep is adequate (e.g., convection). But some classes of problems are at time scale where both come into play, most notably post-glacial rebound.

Newtonian Creep

This is very close on first glance to elasticity, except that we have acquired a time derivative. A Newtonian fluid has a linear relationship between an applied shear stress and the strain rate:

$$\sigma_{xy} = \eta \frac{du_x}{dy} = \eta \frac{d}{dy}\left(\frac{dw_x}{dt}\right) = 2\eta \frac{d}{dt}\left(\frac{1}{2}\frac{dw_x}{dy}\right) = 2\eta \frac{d\varepsilon_{xy}}{dt} \qquad (8)$$

where η is the viscosity of a fluid whose velocity u_x parallels the x axis and varies only with y. As before, we can generalize this to more axes (and include the normal stress terms):

$$\begin{aligned}
\sigma_{xx} &= \lambda_{visc}\theta + 2\eta \frac{d\varepsilon_{xx}}{dt} \\
\sigma_{yy} &= \lambda_{visc}\theta + 2\eta \frac{d\varepsilon_{yy}}{dt} \\
\sigma_{zz} &= \lambda_{visc}\theta + 2\eta \frac{d\varepsilon_{zz}}{dt} \\
\sigma_{xy} &= 2\eta \frac{d\varepsilon_{xy}}{dt} \\
\sigma_{yz} &= 2\eta \frac{d\varepsilon_{yz}}{dt} \\
\sigma_{xz} &= 2\eta \frac{d\varepsilon_{xz}}{dt} \\
\theta &= \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}
\end{aligned} \qquad (9)$$

The λ_{visc} term is often dropped as the compressibility of earth materials is low enough to make it unimportant; when we do this, we are using deviatoric stresses (the normal stresses have had the pressure subtracted out). This is important as the kind of crosstalk between different components seen in elasticity emerges here through the use of deviatoric stresses.

Power-law creep

It would be nice if all materials behaved as a Newtonian fluid; the equations lend themselves to many analytical solutions. When rocks deform by a mechanism termed *diffusion creep*, they are in fact behaving as a Newtonian fluid. However, rocks under upper mantle conditions are found to deform differently, by a second mechanism termed *dislocation creep*. Instead of strain rates being proportional to stresses, we find that relations at depth depend on the rock type involved, the temperature, and the stress difference σ_1 - σ_3. The general form of this is a power-law flow:

$$\dot{\varepsilon} = C(\sigma_1 - \sigma_3)^n e^{-E_a/RT} \tag{4}$$

For olivine (from Goetze, 1978), C is about 7 x 10^4, n is 3, E_a (the activation energy) is 0.52 MJ/mol, and R (the ideal (or universal) gas constant) is 8.317 J K^{-1} $mole^{-1}$. T is in Kelvin. (For dry quartzite, E_a is 0.19 MJ/mol, n is 3, and C is about 5 x 10^{-6}). More recent results tend to put n closer to 3.5 though this has certain theoretical problems. Note that a Newtonian fluid has n=1. In this case, the effective viscosity $\eta_{eff} = C \exp(E_a/RT)$. More generally one can place either part of the stress or strain into the effective viscosity. Here we will place the strain rate into the effective viscosity and get:

$$\begin{aligned} \eta_{eff} &= Be^{E_a/RT}\mathrm{E}^{(1-n)/n} \\ \mathrm{E} &= \sqrt{\tfrac{1}{2}\left[\dot{\varepsilon}_{xx}^2 + \dot{\varepsilon}_{yy}^2 + \dot{\varepsilon}_{zz}^2\right] + \dot{\varepsilon}_{xy}^2 + \dot{\varepsilon}_{yz}^2 + \dot{\varepsilon}_{xz}^2} \end{aligned} \tag{5}$$

B is a constant that depends on mineralogy (and, notably, water content); E is the second invariant of the strain rate tensor (notice the time derivative dots). Effective viscosities are often calculated for a geologically plausible strain rate (e.g., 10^{-15} s^{-1}).

One of the oddities of current modeling of the whole mantle is that despite the evidence for power-law creep in the upper mantle, the bulk of the mantle is presumed to have a Newtonian viscosity. While there are some justifications for this assumption, it may be fair to say that this assumption is as much because the equations are easier than a robust determination that this is, in fact, the viscosity of the deep earth.