Thermal Subsidence of Seafloor

Thermal structure of cooling (or heating) halfspace

We begin by considering the cooling of some freshly minted ocean floor, and we will largely be paralleling discussions developed in Turcotte and Schubert. We will make the problem quite easy by assuming that heat is lost through the top of the oceanic lithosphere and not the sides. We start then with our simple equation for one dimensional change in temperature for material not producing heat internally.

$$\frac{dT}{dt} = \kappa \frac{d^2T}{dz^2} \tag{1}$$

Let us specify the boundary conditions. At the start (t=0), the temperature in the halfspace will be T_a, and at the surface (z=0) it will be T_0. We'll also specify that the temperature at great depth never changes, so at $z=\infty$ we'll force the temperature to be T_a. Given these boundaries, we expect that temperatures will always be between T_0 and T_a, so a dimensionless variable indicating the fraction of the temperature difference will prove helpful:

$$\theta = \frac{T - T_a}{T_0 - T_a} \tag{2}$$

It can be shown that equation (1) is essentially unchanged by the substitution of variables:

$$\frac{d\theta}{dt} = \kappa \frac{d^2\theta}{dz^2} \tag{3}$$

except that the boundary conditions are now θ at the surface is 1, at time 0 it is 0, and at infinity it is 0. The only part of the problem suggesting a length scale is the thermal diffusivity, which is length^2/time, so a practical approach is to try and make the problem totally nondimensional by substituting the dimensionless parameter

$$\eta = \frac{z}{2\sqrt{\kappa t}} \tag{4}$$

We differentiate this with respect to t and z to get the proper terms for (3):

$$\begin{aligned} \frac{\partial\theta}{\partial t} &= \frac{d\theta}{d\eta}\frac{\partial\eta}{\partial t} = \frac{d\theta}{d\eta}\left(-\frac{1}{2}\frac{\eta}{t}\right) \\ \frac{\partial\theta}{\partial z} &= \frac{d\theta}{d\eta}\frac{\partial\eta}{\partial z} = \frac{d\theta}{d\eta}\frac{1}{2\sqrt{\kappa t}} \\ \frac{\partial^2\theta}{\partial z^2} &= \frac{1}{2\sqrt{\kappa t}}\frac{d^2\theta}{d\eta^2}\frac{\partial\eta}{\partial z} = \frac{1}{4\kappa t}\frac{d^2\theta}{d\eta^2} \end{aligned} \tag{5}$$

which, when placed back into (3) yields

$$-\eta\frac{d\theta}{d\eta} = \frac{1}{2}\frac{d^2\theta}{d\eta^2} \tag{6}$$

and the boundary conditions have reduced to $\theta(\infty)=0$ and $\theta(0)=1$. We first solve for $\frac{d\theta}{d\eta}$ which we will term ϕ, so (6) becomes

$$-\eta\phi = \frac{1}{2}\frac{d\phi}{d\eta} \tag{7}$$

which is easily integrated to find that

$$-\eta^2 = \ln\phi - \ln c_1 \tag{8}$$

where c_1 is a constant of integration. If we take the exponential of both sides we get

$$\phi = c_1 e^{-\eta^2} = \frac{d\theta}{d\eta} \tag{9}$$

which can also be integrated to solve for θ:

$$\theta = c_1\int_0^\eta e^{-\eta'^2}\,d\eta' + c_2 \tag{10}$$

Applying our boundary conditions, when $\eta = 0$ θ should be 1, and as the integral from 0 to 0 is 0, c_2 must be 1. When $\eta = \infty$ the integral becomes one commonly known to be $\frac{\sqrt{\pi}}{2}$ and so therefore c_1 must be $-\frac{2}{\sqrt{\pi}}$, which leads us to the solution

$$\theta = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta'^2}\,d\eta' \tag{11}$$

This is a common enough integral that it has it's own name, the error function, $\mathrm{erf}(\eta)$ and so we rewrite our solution as

$$\theta = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}\,\eta \tag{12}$$

where erfc is the complementary error function. We can substitute our original variables back in to get

$$\frac{T - T_a}{T_0 - T_a} = \operatorname{erfc}\frac{z}{2\sqrt{\kappa t}} \tag{13}$$

What this tells us is that the depth of an isotherm increases with $\sqrt{\kappa t}$. A simple rearrangement of (13) is also handy:

$$1 - \frac{T - T_a}{T_0 - T_a} = \frac{T_0 - T}{T_0 - T_a} = \frac{T - T_0}{T_a - T_0} = \operatorname{erf}\frac{z}{2\sqrt{\kappa t}} \tag{14}$$

This is, in essence, the solution that Lord Kelvin used to argue for the young age of the Earth (differentiating with respect to depth *z* gets an equation for thermal gradient with depth over time).

Subsidence of seafloor

If we assume that all variations in topography of oceanic lithosphere are due to temperature variations (also called Pratt isostasy sometimes, after an early advocate of supporting topography through variations in densities), we can directly use our analysis to get at variations in topography. This is not a bad assumption, for most seafloor crust is generated very near the ridgecrest and is not modified much over its whole history. In fact, by removing the thermal variations you can see these other effects more clearly. So let us assume local isostasy. The condition for local isostasy is that the pressure at some reference depth is the same everywhere. The pressure in turn is the integral of the weight of all the material above:

$$P_c = \int_0^{z_c} \rho g\, dz = dg\rho_w + \int_0^{z_c'} \rho g\, dz' \tag{15}$$

where P_c is the pressure at the depth of compensation, which is z_c below sea level or z'_c below the seafloor. We have *d* as the depth of the water. If we now set the pressure under a column at the midocean ridge at a water depth of *d* equal that under a column elsewhere at depth *w*+*d* of age *t*, we get

$$\begin{gathered} d_{ridge} g\rho_w + \int_d^{z_c} \rho g\, dz = \left(d_{ridge} + w\right) g\rho_w + \int_{d_{ridge}+w}^{z_c} \rho g\, dz \\ \int_{d_{ridge}}^{d_{ridge}+w} \rho_a dz + \int_{d_{ridge}+w}^{z_c} \rho_a dz = w\rho_w + \int_{d_{ridge}+w}^{z_c} \rho(z)\, dz \\ w\left(\rho_a - \rho_w\right) = \int_{d_{ridge}+w}^{z_c} \left(\rho(z) - \rho_a\right) dz \end{gathered} \tag{16}$$

We can use the coefficient of thermal expansion to relate our density to the temperature, $\rho - \rho_a = -\rho_a \alpha\left(T - T_a\right)$, and then use the temperature from (13), converting our limits of integration to now be downward from the seafloor of the old ocean floor to the depth of compensation under that seafloor:

$$\begin{aligned} w\left(\rho_a - \rho_w\right) &= \int_0^{z'_c} -\rho_a \alpha\left(T_0 - T_a\right) \operatorname{erfc} \frac{z'}{2\sqrt{\kappa t}} dz' \\ &= \rho_a \alpha\left(T_a - T_0\right) \int_0^{\infty} \operatorname{erfc} \frac{z'}{2\sqrt{\kappa t}} dz' \end{aligned} \tag{17}$$

We again allowed the integral to go to infinity because the temperatures converge below z'_c. We repeat the substitution used in (4) so that the integral is more simply dealt with:

$$\boxed{\begin{aligned} w &= \frac{\rho_a \alpha\left(T_a - T_0\right)}{\rho_a - \rho_w} 2\sqrt{\kappa t} \int_0^{\infty} \operatorname{erfc} \eta \, d\eta \\ &= \frac{2\rho_a \alpha\left(T_a - T_0\right)}{\rho_a - \rho_w} \sqrt{\frac{\kappa t}{\pi}} \end{aligned} \qquad (18)}$$

where we have again used a standard result for the integral of the complementary error function. This is the desired result that expresses the depth of the seafloor relative to the ridge as the square root of the age of that sea floor. If spreading is constant, then the time can be replaced by the distance from the ridge over the half spreading rate.