

# The CDF and Conditional Probability

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# 1 Conditional CDFs

We are used to playing with conditional distributions in terms of the PDF, but there is more than one way to skin a cat. Here we will work with conditional distributions in terms of the CDF.

To begin, the general form of a conditional CDF is as follows<sup>1</sup>:

$$F_{XY}(x | y) = Pr[X \leq x | Y \leq y] = \frac{F_{XY}(x,y)}{F_Y(y)}$$

Where  $F_{XY}(x, y)$  denotes the joint CDF and  $F_Y(y)$  is the marginal CDF of the random variable  $Y$ . Please note, this general formula can be applied to continuous, discrete and mixture distributions. Dividing the joint CDF by the marginal CDF allows us to normalize our conditional distribution so that it maintains the necessary properties of the CDF only in terms of the variable we are not conditioning upon. For example, if we looked at the formula above, this particular conditional CDF only maintains the properties of the CDF in terms of  $X$ .

Recall the properties of the CDF:

- 1)  $\lim_{x,y \rightarrow -\infty} F_{XY}(x, y) = 0$  and  $\lim_{x,y \rightarrow \infty} F_{XY}(x, y) = 1$
- 2)  $F_{XY}(x, y)$  is a monotone, nondecreasing function for both  $X$  and  $Y$
- 3)  $F_{XY}(x, y)$  is continuous from the right for both  $X$  and  $Y$

Remember, the conditional CDF retains all the same properties as the CDF for only the random variable you are not conditioning upon. Later on, we will use an example to demonstrate that these properties hold for conditional CDFs.

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<sup>1</sup>Kingsbury, Nick. Joint and Conditional cdfs and pdfs. Connexions. 7 June 2005 <<http://cnx.org/content/m10986/2.8/>>.

## 1.1 Continuous Conditional CDFs

We will begin by looking at the case where the variables are jointly continuous. In terms of the CDF there is one primary method we use; it is direct, in so far that we go from a CDF to a conditional CDF.

There is an alternative method that involves taking the conditional PDF and transforming it into a conditional CDF. We will focus on the first method, as it better helps us understand the conditional CDF as its own beast, independent of the PDF.

### 1.1.1 Direct CDF Method

Transforming a joint CDF into a conditional CDF requires that we figure out what the marginal CDF is. The marginal CDF of X can be found by taking the limit of the joint CDF as Y approaches infinity:

$$F_X(x) = \lim_{y \rightarrow \infty} F_{XY}(x, y)$$

And conversely for the marginal CDF of Y:

$$F_Y(y) = \lim_{x \rightarrow \infty} F_{XY}(x, y)$$

Now that we know how to find the marginal CDF, we can use the following equation to determine the conditional CDF,  $F_{XY}(x | y)$ :

$$F_{XY}(x | y) = Pr[X \leq x | Y \leq y] = \frac{F_{XY}(x, y)}{F_Y(y)}$$

### 1.1.2 Moments in the Conditional CDF

It may prove useful to obtain the first moment of a conditional CDF. That is, what is the expectation of a random variable  $X$  given that we know that the random variable  $Y$  is less than a given value. Notationally, we are interested in  $E[X | Y \leq y]$ . To do this, first recall the formula for the mean of the CDF:

$$E[X] = \int_0^\infty 1 - F_X(x) dx - \int_{-\infty}^0 F_X(x) dx$$

So our desired expectation will follow a near identical form:

$$E[X | Y \leq y] = \int_0^\infty 1 - \frac{F_{XY}(x,y)}{F_Y(y)} dx - \int_{-\infty}^0 \frac{F_{XY}(x,y)}{F_Y(y)} dx$$

What you may notice here is that we are only integrating with respect to  $X$ . This leads us to an interesting point about this expectation: it will vary with  $Y$ . This makes intuitive sense, as you would expect that  $E[X | Y \leq a] \neq E[X | Y \leq b]$ .

### 1.1.3 Direct CDF Method Examples

**Example 1:** Taking a new spin on an example from the lecture on joint density functions, where we were given the following CDF:

$$F_{XY}(x, y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ .5xy(x + y) & \text{if } 0 \leq x < 1 \text{ and } 0 \leq y < 1 \\ 1 & \text{if } x \geq 1 \text{ and } y \geq 1 \\ .5x(x + 1) & \text{if } 0 \leq x < 1 \text{ and } y \geq 1 \\ .5y(1 + y) & \text{if } x \geq 1 \text{ and } 0 \leq y < 1 \end{cases}$$

First we will generally solve for the conditional CDF, which we can then use to find our probabilities of interest.

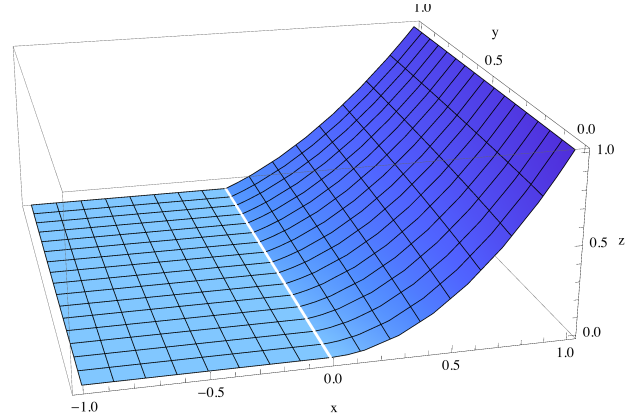
To solve this problem we need the joint CDF (which we have) and the marginal CDF (which we do not have, yet). Specifically, we need the marginal CDF of Y:

$$F_Y(y) = \lim_{x \rightarrow \infty} F_{XY}(x, y) = \begin{cases} 0 & \text{if } y < 0 \\ .5y(1 + y) & \text{if } 0 \leq y < 1 \\ 1 & \text{if } y \geq 1 \end{cases}$$

Deriving the conditional CDF:

$$F_{XY}(x | y) = \frac{F_{XY}(x, y)}{F_Y(y)} = \frac{.5xy(x+y)}{.5y(1+y)}$$

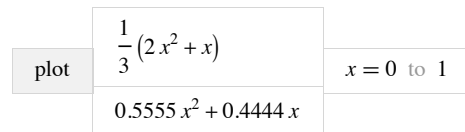
Plotting our conditional CDF,  $F_{XY}(x | y) = \frac{F_{XY}(x,y)}{F_Y(y)} = \frac{.5xy(x+y)}{.5y(1+y)}$ :



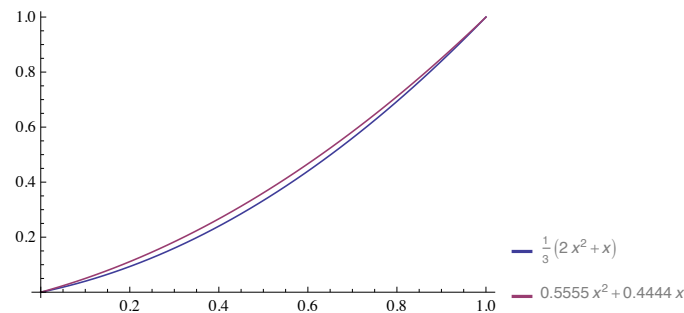
This graph helps us to see that our conditional CDF maintains all the necessary properties of the CDF. Overall, we can see the right continuity of  $X$ , the limit of  $X$  as it approaches its lower bound of 0 is 0, the limit of  $X$  as it approaches its upper bound of 1 is 1 and the conditional CDF is nondecreasing for  $X$ . It's all there.

At first glance, one might think that the variables are independent, since it does not appear that variation in  $Y$  affects the shape of the plot. This is incorrect, varying  $Y$  does alter the distribution, just very minimally. To show this, examine the following plot:

Input interpretation:

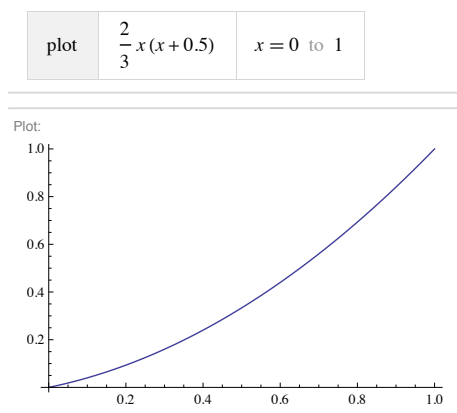


Plot:



What we've done here is fix the value of  $Y$  for our conditional CDF,  $\frac{F_{XY}(x,y)}{F_Y(y)}$ , at two values of  $Y$ , 0.5 and 0.8. What you see on the plot is  $\frac{F_{XY}(x,.5)}{F_Y(.5)}$ , denoted by the blue line and  $\frac{F_{XY}(x,.8)}{F_Y(.8)}$ , denoted by the pinkish line. These two lines represent slices of the conditional distribution plotted above. Clearly, the slices are not identical, but very, very similar.

Lets try and find  $\Pr[X \leq x | Y \leq .5]$ . In this case, we are only looking at a specific slice of the graph from above. Recall that  $F_Y(.5)$  is equivalent to the probability that  $Y \leq .5$ . With this in mind, we take the conditional CDF,  $\frac{F_{XY}(x,y)}{F_Y(y)} = \frac{.5xy(x+y)}{.5y(1+y)}$ , and fix  $Y = .5$  to arrive at the following plot:



Just looking at the graph above, you should be able to see that it maintains the properties of the CDF.

Say we were looking for the probability that  $X \leq .75$  given that we know  $Y \leq .5$ . To solve this, we would proceed as follows:

$$\Pr[X \leq .75 | Y \leq .5] = F(.75 | .5)$$

And we know that:

$$F(.75 | Y \leq .5) = \frac{F(.75,.5)}{F_Y(.5)} = \frac{(.5)(.75)(.5)(.75+.5)}{(.5)(.5)(1+.5)} = .625$$

Examining the 2D plot above and tracing up from .75 on the horizontal axis gives us a point that looks pretty close to .625 on the vertical axis, which is what we are looking for. Using our conditional CDF, we have shown there is a 62.5% chance that X is less than or equal to .75, given that Y is less than or equal to .5.

For fun, lets find the first moment of this conditional CDF. To start, we will derive the general form,  $E[X | Y \leq y]$ .

$$E[X | Y \leq y] = \int_0^\infty 1 - \frac{.5xy(x+y)}{.5y(1+y)} dx - \int_{-\infty}^0 \frac{.5xy(x+y)}{.5y(1+y)} dx$$

$$E[X | Y \leq y] = \int_0^1 1 - \frac{.5xy(x+y)}{.5y(1+y)} dx - 0$$

$$E[X | Y \leq y] = 1 - \left(\frac{1}{y+1}\right)\left(\frac{1}{3} + \frac{y}{2}\right)$$

As you can see, the expectation of  $X$  varies with  $Y$ . From here, one can just plug in values for  $y$  as follows:

$$E[X | Y \leq .1] = .6515$$

$$E[X | Y \leq .5] = .6111$$

$$E[X | Y \leq .9] = .5877$$

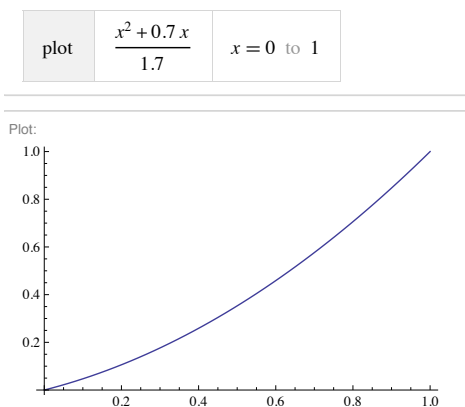
See, tons of fun.

**Example 2:** Sometimes we want to find the probability that our random variable,  $X$ , is within a certain range of values, given some range of  $Y$ . Using the same CDF as above, we could ask, what is the probability that  $X$  is between .2 and .4, given  $Y \leq .7$ . Notationally this is going to be  $Pr[.2 \leq X \leq .4 | Y \leq .7]$ .

This can be found by taking the conditional CDF for  $X \leq .4$  and subtracting from it the conditional CDF for  $X \leq .2$  as seen here:

$$Pr[X \leq .4 | Y \leq .7] - Pr[X \leq .2 | Y \leq .7]$$

Once again, recall that  $F_Y(.7)$  is equivalent to the probability that  $Y \leq .7$ . We can create a plot by fixing  $Y = .7$  in our conditional CDF,  $\frac{F_{XY}(x,y)}{F_Y(y)} = \frac{.5xy(x+y)}{.5y(1+y)}$ , to provide some intuition:



We will begin by calculating the probability that  $X$  is at most .4 given  $Y$  is at most .7:

$$Pr[X \leq .4 | Y \leq .7] = F(.4|.7) = \frac{F_{XY}(.4,.7)}{F_Y(.7)} = \frac{.5(.4)(.7)(.4+.7)}{.5(.7)(1+.7)} = \frac{.154}{.595} = .2588$$

Next will find the probability that  $X$  is at most .2 given  $Y$  is at most .7:

$$Pr[X \leq .2 | Y \leq .7] = F[.2|.7] = \frac{F_{XY}(.2,.7)}{F_Y(.7)} = \frac{.5(.2)(.7)(.2+.7)}{.5(.7)(1+.7)} = \frac{.063}{.595} = .10588$$

Combining these results will yield our answer:

$$Pr[.2 \leq X \leq .4 | Y \leq .7] = .2588 - .10588 = .1529$$

In words, there is a 15.29% chance that  $X$  is between .2 and .4, given  $Y$  is less than or equal to .7.

**Method 2:** Transforming a conditional PDF to a conditional CDF is done in essentially the same manner as transforming a regular PDF to a CDF. For continuous conditional PDFs, we utilize integration.

#### 1.1.4 Method 2 Example

Using a joint density function that we just made up:

$$f_{XY}(x, y) = 2x^2y + 2y^2 \text{ if } 0 \leq x, y \leq 1$$

$$f_{XY}(x, y) = 0 \text{ otherwise}$$

Since we desire  $f(x | y)$ , we must first derive the marginal PDF of  $Y$ :

$$f_Y(y) = \int_0^1 2x^2y + 2y^2 dx = 2y^2 + \frac{2}{3}y$$

We know from our initial lecture on joint density functions that a conditional joint density function takes the following form:

$$f(x | y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2x^2y+2y^2}{2y^2+\frac{2}{3}y} = \frac{x^2+y}{y+\frac{1}{3}}$$

Applying our formula from MGB we can convert the conditional PDF to a conditional CDF through integration:

$$F_{X|Y}(x | y) = \int_{-\infty}^x f_{x|y}(z | y) dz$$

$$F_{X|Y}(x | y) = \int_0^x \frac{z^2+y}{y+\frac{1}{3}} dz$$

$$F_{X|Y}(x | y) = \frac{x^3+3xy}{3y+1}$$

Therefore, our conditional CDF,  $F_{X|Y}(x | y) = \frac{x^3+3xy}{3y+1}$

Say we were asked to find the probability that  $X \leq .4$  given that we know  $Y \leq .7$ . We can employ our conditional CDF to determine the answer:

$$Pr[X \leq .4 | Y \leq .7] \equiv F_{X|Y} [.4, .7] = \frac{.4^3+3(.4)(.7)}{3(.7)+1} = \frac{.064+.84}{3.1} = .2916$$

This result tells us that if we know  $Y \leq .7$ , then there is approximately a 29.17% chance that  $X \leq .4$ .

### 1.1.5 An Application of Conditional CDFs - Mixture Distributions

So let's say we want to study some population of two distinct groups. As discussed in a previous lecture, we know that we would want to create a mixture distribution.

Lets assume that 90% of the population can be modeled by the following distribution:

$$h(x, y) = x + y$$

And we know the remaining 10% of the population can be modeled as follows:

$$g(x, y) = 2x^2y + 2y^2$$

So we know our mixture distribution would look like this:

$$f(x, y) = (.9)(x + y) + (.1)(2x^2y + 2y^2)$$

Where  $X$  and  $Y$  are bounded between zero and one for  $h$ ,  $g$  and  $f$ . Now we are interested in finding the conditional CDF of this distribution,  $F_{X|Y}(x | y)$ . What we want to show here is that:

$$F_{X|Y}(x | y) = (.9)H_{X|Y}(x | y) + (.1)G_{X|Y}(x | y)$$
$$F_{X|Y}(x | y) = (.9)\frac{.5xy(x+y)}{.5y(1+y)} + (.1)\frac{x^3+3xy}{3y+1}$$

So lets compute an arbitrary value and see what we find. For fun, lets find  $Pr[X \leq .5 | Y \leq .5]$ , or notationally,  $F_{X|Y}(.5 | .5)$ :

$$F_{X|Y}(.5 | .5) = (.9)H_{X|Y}(.5 | .5) + (.1)G_{X|Y}(.5 | .5)$$
$$F_{X|Y}(.5 | .5) = (.9)\frac{.125}{.375} + (.1)\frac{.875}{2.5}$$
$$F_{X|Y}(.5 | .5) = .3 + .035 = .335$$

Well, the result is less than 1, which we want, but how do we know that it is right? Let's try going through all of our steps to create our mixture conditional CDF.

Starting from scratch, we can multiply out the probabilities into the distribution to get one big joint PDF:

$$f(x, y) = (.9)(x + y) + (.1)(2x^2y + 2y^2) = .9x + .9y + .2x^2y + .2y^2$$

We know from above that in order to arrive at the conditional CDF,  $F_{X|Y}(x | y)$ , we need to find both the joint CDF of  $X$  and  $Y$ ,  $F_{XY}(x, y)$ , and the marginal CDF of  $Y$ ,  $F_Y(y)$ :

$$F_{XY}(x, y) = \int_0^1 \int_0^1 (.9a + .9b + .2a^2b + .2b^2) da db$$

$$F_{XY}(x, y) = xy\left(\frac{1}{30}x^2y + .45x + \frac{y^2}{15} + .45y\right)$$

$$F_Y(y) = \lim_{x \rightarrow 1} F_{XY}(x, y) = y\left(\frac{y}{30} + .45 + \frac{y^2}{15} + .45y\right)$$

Therefore, our conditional CDF is as follows:

$$F_{X|Y}(x | y) = \frac{F_{XY}(x, y)}{F_Y(y)} = \frac{xy\left(\frac{1}{30}x^2y + .45x + \frac{y^2}{15} + .45y\right)}{y\left(\frac{y}{30} + .45 + \frac{y^2}{15} + .45y\right)}$$

So now we can compute our desired probability in our alternate notation,  $Pr[X \leq .5 | Y \leq .5]$ :

$$F_{X|Y}(.5 | .5) = \frac{F_{XY}(.5, .5)}{F_Y(.5)} = \frac{.1177}{.3511} = .335$$

This is the same answer as before, but the steps to get there are much uglier. The good thing is, we've shown in this example that we can apply our conditional CDF techniques to mixture distributions.

## 1.2 Discrete Conditional CDFs

### 1.2.1 Derivation

So far, we've only been discussing conditional CDFs in the context of jointly continuous random variables. We will now discuss the role of conditional CDFs among discrete random variables. While one can certainly apply the same formula initially presented in these notes, it proves to be easier to use an alternative method. This alternative method involves taking the conditional PDF and transforming it into a conditional CDF.

**Method:** Transforming a conditional PDF to a conditional CDF is done in essentially the same manner as transforming a regular PDF to a CDF. For discrete conditional PDFs, we use summation.

Assuming  $X$  and  $Y$  are two jointly discrete random variables, the conditional cumulative distribution of  $Y$  given that  $X = c$  is given by:

$$F_{Y|X}(y | c) = P[Y \leq y | X = c]$$

MGB defines the conditional CDF as follows:

$$F_{Y|X}(y | c) = \sum_{j: y_j \leq y} f_{Y|X}(y_j | c)$$

To demonstrate this formula, we will extend an example from MGB, page 145.

### 1.2.2 Discrete CDF Example

We are rolling two tetrahedra (4 sided die). Essentially, tetrahedra are pyramids with 3 sides and a base. We are interested in the downturned face since it is the only unique result, as the 3 remaining faces of the tetrahedra are sides of the pyramid. The random variable  $X$  represents the downturned face of the first die and  $Y$  represents the downturned face of the die with the largest downturned value. This can be a bit tricky so we will provide some examples.

**Case 1:** If the downturned face of the first die is 2 and the second die is 4,  $X = 2$  since it depends only on the first die,  $Y = 4$  since  $Y$  takes on the higher value of the dice.

**Case 2:** If the downturned face of the first die is 3 and the second die is 2,  $X = 3$  since it depends only on the first die,  $Y = 3$  since  $Y$  takes on the higher value of the dice. You get the picture.

One question we could ask is what is the probability that  $Y \leq 3$  given that  $X = 3$ .

Essentially, we need:  $F_{Y|X}(3 | 3) = \sum_{j:y_j \leq 3} f_{Y|X}(y_j | 3)$ , and we can solve this using MGB's definition, provided we know the conditional PDFs.

As it turns out, there are 3 conditional PDFs we need:  $f_{Y|X}(1 | 3)$ ,  $f_{Y|X}(2 | 3)$ , and  $f_{Y|X}(3 | 3)$

It follows trivially that  $f_{Y|X}(1 | 3) = f_{Y|X}(2 | 3) = 0$ . This lies in the fact that  $Y$  must always be at least as large as  $X$ . Hence, there is a zero percent chance that  $Y$  is less than  $X$  since  $Y$  is restricted to be greater than or equal to  $X$ .

So now we must find  $f_{Y|X}(3 | 3)$ :

$$f_{Y|X}(3 | 3) = \frac{f_{XY}(3,3)}{f_X(3)}$$

To determine the value of  $f_{XY}(3, 3)$ : Since there are 4 possible outcomes when tossing a tetrahedra, the likelihood that  $X = 3$  is  $\frac{1}{4}$ , in order for Y to be 3, if the first roll is 3, the second roll can be 1,2 or 3, and that the probability that the second roll takes on those values is  $\frac{3}{4}$ . This implies that:

$$f_{XY}(3, 3) = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \frac{3}{16}.$$
$$f_X(3) = \frac{1}{4}$$

Now that we have what we're looking for, we can find the answer:

$$f_{Y|X}(3 | 3) = \frac{f_{XY}(3,3)}{f_X(3)} = \frac{\frac{3}{16}}{\frac{1}{4}} = \frac{3}{4}$$

Summing our three values yields the answer in terms of the discrete conditional CDF:

$$F_{Y|X}(3 | 3) = f_{Y|X}(1 | 3) + f_{Y|X}(2 | 3) + f_{Y|X}(3 | 3) = 0 + 0 + \frac{3}{4} = \frac{3}{4}$$

## 2 Review Questions

### 2.1 Discrete Case - Review Questions

1) Given the following joint PDF:

$$p_{XY}(x, y) = \frac{1}{15}(x + y), \text{ for } x = 0, 1, 2 \text{ } y = 1, 2$$
$$p(x, y) = 0, \text{ otherwise}$$

- Derive the joint CDF
- Find  $Pr(X \leq 1 | Y = 2)$

**Answer:** a.  $P_{XY}(x, y) = \sum_{x=0}^2 \sum_{y=1}^2 \frac{1}{15}(x + y)$

b.  $Pr(X \leq 1 | Y = 2) = \sum_{x=0}^1 \frac{1}{15}(x + 2) = \frac{2}{15} + \frac{3}{15} = \frac{1}{3}$

2) Observe the following table representing a discrete joint PDF:

Y / X	1	2	3	4
1	$\frac{1}{40}$	$\frac{3}{40}$	$\frac{1}{40}$	$\frac{4}{40}$
2	$\frac{1}{40}$	$\frac{5}{40}$	$\frac{1}{40}$	$\frac{1}{40}$
3	$\frac{3}{40}$	$\frac{1}{40}$	$\frac{3}{40}$	$\frac{4}{40}$
4	0	$\frac{2}{40}$	$\frac{3}{40}$	$\frac{3}{40}$
5	0	$\frac{3}{40}$	$\frac{1}{40}$	0

First, verify that this is a proper PDF.

Second, find  $P(X \leq 3 | Y = 4)$

Third, find  $P(X \leq 2 | Y \leq 3)$

**Answer:** Part one is simple. All one must do is sum all the values and ensure they sum to one. Effectively you are verifying that

$$\sum_{x=1}^4 \sum_{y=1}^5 P_{XY}(x, y) = 1.$$

Part two is a bit more complex. You are effectively summing up the values of X from 1 to 3 when Y is equal to 4. Therefore,

$$P(X \leq 3 | Y = 4) = 0 + \frac{2}{40} + \frac{3}{40} = \frac{5}{40} = \frac{1}{8}$$

Part three follows similarly to part two, but this time you sum all the values of X from 1 to 2 and Y from 1 to 3. Therefore,

$$P(X \leq 2 | Y \leq 3) = \frac{1}{40} + \frac{1}{40} + \frac{3}{40} + \frac{3}{40} + \frac{5}{40} + \frac{1}{40} = \frac{14}{40} = \frac{7}{20}$$

3) Let A and J denote the number of aces and jacks in a given poker hand, respectively. If you have already been dealt two jacks, what is the probability that you will receive *at least* two aces in your hand?

**Answer:** We are looking for a discrete conditional probability that utilizes the CDF; that is, we want  $\Pr(A \geq 2 \mid J = 2)$ . If we want to use the CDF to find the answer, recall:  $\Pr(X \leq a) = 1 - F(a)$ , or if using our notation,  $\Pr(A \geq 2 \mid J = 2) = 1 - [\Pr(A \leq 1 \mid J = 2)]$ .

Recalling MGB's definition of discrete conditional probability, where:

$$P_{Y|X}(c \mid x) = \sum_{y \leq c} p_{Y|X}(y \mid x) = \sum_{\min y}^c \frac{p_{XY}(x,y)}{p_X(x)}$$

Thus, we must find both the joint PDF  $[p_{AJ}(a, j)]$  and marginal PDF of Jacks  $[p_J(j)]$  to solve this problem.

$$p_{AJ}(a, j) = \frac{\binom{4}{j} \binom{4}{a} \binom{44}{5-a-j}}{\binom{52}{5}}$$

**For:**

$$a = 0, 1, 2, 3, 4$$

$$j = 0, 1, 2, 3, 4$$

$$a + j \leq 5$$

$$p_J(j) = \frac{\binom{4}{j} \binom{48}{3}}{\binom{52}{5}} \text{ For } j = 0, 1, 2, 3, 4$$

To arrive at our desired probability,  $\Pr(A \geq 2 \mid J = 2)$ , we must find:

$$\Pr(A \leq 1 \mid J = 2) \equiv \sum_{a=0}^1 \frac{p_{AJ}(a,2)}{p_J(2)}$$

Through a little algebra:

$$\sum_{a=0}^1 \frac{p_{AJ}(a,2)}{p_J(2)} = \sum_{a=0}^1 \frac{\binom{4}{a} \binom{44}{3-a}}{\binom{48}{3}} \approx .9845$$

$$\Rightarrow \Pr(A \geq 2 \mid J = 2) \approx 1 - .9845 = \mathbf{.0155}$$

## 2.2 Continuous Case - Review Questions

1) Create a bivariate CDF and verify it has the properties of a CDF. Solve for the conditional CDF,  $F_{X|Y}(X | Y = c)$ .

**Answer:** It will depend on the CDF you choose.

2) Given the following joint CDF:

$$F(x, y) = \begin{cases} 0 & \text{if } x < 0 \text{ and } y < 0 \\ (\frac{1}{2} + \frac{1}{2}y^2) & \text{if } x > 1 \text{ and } 0 \leq y \leq 1 \\ (\frac{1}{2}x^2 + \frac{1}{2}y^2) & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ (\frac{1}{2}x^2 + \frac{1}{2}) & \text{if } 0 \leq x \leq 1 \text{ and } y > 1 \\ 1 & \text{if } x > 1 \text{ and } y > 1 \end{cases}$$

Use the conditional CDF to determine the probability that X is less than or equal to given Y is less than or equal to .25.

**Answer:** We want to solve for  $Pr(X \leq .8 | Y \leq .25)$ :

$$Pr(X \leq .8 | Y \leq .25) = F_{X|Y}(.8 | .25) = \frac{F_{XY}(.8, .25)}{F_Y(.25)}$$

$$\frac{F_{XY}(.8, .25)}{F_Y(.25)} = \frac{(.5)(.8^2) + (.5)(.25^2)}{(.5) + (.5)(.25)} = \frac{.35125}{.53125} = .6611$$

Thus, our answer is that there is approximately a 66.11% chance that X is less than or equal to .8 given that Y is less than or equal to .25.

3) Given the CDF:

$$F(x, y) = \begin{cases} 0 & \text{if } X < 0 \text{ or } Y < 0 \\ (\frac{1}{3}y + \frac{2}{3}) & \text{if } X > 1 \text{ and } 0 \leq Y \leq 1 \\ (\frac{1}{3}xy + \frac{2}{3}x) & \text{if } 0 \leq X \leq 1 \text{ and } 0 \leq Y \leq 1 \\ (\frac{1}{3}x + \frac{2}{3}x) & \text{if } 0 \leq X \leq 1 \text{ and } Y > 1 \\ 1 & \text{if } X, Y > 1 \end{cases}$$

What is  $Pr[.5 \leq X \leq .9 | Y \leq .4]$ ?

**Answer:** Solve it as the difference between two conditional CDFs.

$$Pr[X \leq .9 | Y \leq .4] = \frac{F_{XY}(.9,.4)}{F_Y(.4)} = \frac{(\frac{1}{3})(.9)(.4) + (\frac{2}{3})(.9)}{\frac{1}{3}(.4) + \frac{2}{3}} = \frac{.72}{.8} = .9$$

$$Pr[X \leq .5 | Y \leq .4] = \frac{F_{XY}(.5,.4)}{F_Y(.4)} = \frac{\frac{1}{3}(.5)(.4) + \frac{2}{3}(.5)}{\frac{1}{3}(.4) + \frac{2}{3}} = \frac{.4}{.8} = .5$$

As  $Pr[.5 \leq X \leq .9 | Y \leq .4] = Pr[X \leq .9 | Y \leq .4] - Pr[X \leq .5 | Y \leq .4]$ ,  
we get  $.9 - .5 = .4$

So there is a 40% chance that X is between .5 and .9, given that Y is less than or equal to .4.

### 3 Works Cited

- Kingsbury, Nick. Joint and Conditional cdfs and pdfs. Connexions. 7 June 2005 <<http://cnx.org/content/m10986/2.8/>>.
- Leon-Garcia, Alberto, and Alberto Leon-Garcia. *Probability, Statistics, and Random Processes for Electrical Engineering*. Upper Saddle River, NJ: Pearson/Prentice Hall, 2008.
- Mood, Alexander McFarlane, Franklin A. Graybill, and Duane C. Boes. *Introduction to the Theory of Statistics*. New York: McGraw-Hill, 1973.