A Note on how to randomly sample from the $f(x)$ distribution

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Many software packages have commands to draw random samples from the standard normal distribution and the uniform distribution on the 0 to 1 (hereafter the unit uniform).

However, commands and software do not generally exist for taking random samples from other distributions.

Assume the problem is take a random sample from the $f_X(x)$ distribution where one has the ability (e.g. Mathematica) to take a random sample from the uniform distribution on the 0 - 1 interval.

Let $F_X(X)$ denote the cdf of $f_X(X)$

One can show that one can obtain a random draw from $f_X(X)$ by obtaining a random draw from the unit uniform distribution, $f_U(U)$, then plugging that draw into $F_X^{-1}(U)$ where $F_X^{-1}(.)$ is the inverse of $F_X(X)$. That is, $F_X^{-1}(U)$ is $h = F_X(X)$ solved for X and then evaluated at U.

Example 1:

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Take a random sample from the (negative) exponential distribution (MGB p. 112). If X has a negative exponential distribution

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & if \quad 0 \leq x \leq \infty \\ 0 & if \quad x < 0 \end{cases}$$

where $\lambda > 0$. The corresponding cdf is

$$h = F_X(x) = 1 - e^{-\lambda x}$$

Solve $h = 1 - e^{-\lambda x}$ for x, Solution is: $\left\{x = -\frac{\ln(-h+1)}{\lambda}\right\}$. So

$$F_X^{-1}(u) = -\frac{\ln(-u+1)}{\lambda}$$

In words, if u is a random draw from the unit uniform, then $-\frac{\ln(-u+1)}{\lambda}$ is a random draw from the exponential distribution.

One draws a random sample of N observations from the unit uniform, then converts each using the formula $-\frac{\ln(-u+1)}{\lambda}$

Example 2:

I often use this "rule" to take to draw a random sample from a population with an Extreme Value distribution. The cdf for the Extreme Value distribtuion is

$$F_X(x) = \Pr(X \leq x) = \exp^{-e^{-x}}$$

That is $h = \exp^{-e^{-x}} \Rightarrow \ln h = -e^{-x} \Rightarrow -\ln h = e^{-x} \Rightarrow \ln(-\ln h) = -x \Rightarrow$

$$x = -\ln(-\ln h)$$

So, if if u is a random draw from the unit uniform, then $x = -\ln(-\ln u)$ is a random draw from the Extreme Value distribution.

Some Background

This method for taking random samples is an application of Theorem 12 (sec 5.2) of MGB page 202. It is called the Probability Integral Transform.

Theorem 12 says "If X is a random variable with a continuous cummulative distribution function $F_X(x)$, then $U = F_X(X)$ is uniformly distribution over the interval $(0,1)$. Conversely, if U is uniformly distributed on the interval $(0,1)$, then $X = F_X^{-1}(U)$ has the cummulative distribution function $F_X(.)$.