

Joint Density Functions, Marginal Density Functions, Conditional Density Functions, Expectations and Independence

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Define the joint density

Consider the joint density function

$$f(x,y)$$

where $f(x,y) \geq 0 \forall x$ and y , $-\infty < x, y < +\infty$, and

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) dx dy = 1.$$

Any function that fulfills these properties is a joint density function.

Make up a two variable joint density function and demonstrate that it is a density function.

First example:

Is the following function a joint density function?

$$f(x,y) = \begin{cases} x + y & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

That is, does the function have the following properties: $f(x,y) \geq 0 \forall x, y$ and $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) dx dy = 1$?

Yes to the first. Check the second.

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) dx dy &= \int_0^1 \int_0^1 (x + y) dx dy \\ &= \int_0^1 [.5x^2 + xy]_0^1 dy \\ &= \int_0^1 (.5 + y) dy \\ &= .5y + .5y^2 \Big|_0^1 \\ &= .5 + .5 = 1 \end{aligned}$$

So yes, it is a joint density function.

Example 2:

I started by assuming $h(x,y) = x^a y^{1-a}$ if $0 \leq x \leq 1$ and $0 \leq y \leq 1$, and 0 otherwise, where $0 < a < 1$ - sort of a Cobb-Douglas density function. This function never takes negative values, so fulfills the first property of a density function. I determined that

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(x,y) dx dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^a y^{1-a} dx dy \\ &= \int_0^1 \int_0^1 x^a y^{1-a} dx dy \\ &= 1/(2 + a - a^2) \neq 1 \end{aligned}$$

so $h(x,y)$ is not a density function. However one can easily turn it into a density function by multiplying $h(x,y)$ by $(2 + a - a^2)$ to obtain the density function $f(x,y) = (2 + a - a^2)x^a y^{1-a}$ if $0 \leq x \leq 1$ and $0 \leq y \leq 1$, and 0 otherwise, where $0 < a < 1$.

Example 3

Is the following function a joint density function?

$$f(x,y) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

That is, does the function have the following properties: $f(x,y) \geq 0 \forall x,y$ and $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) dx dy = 1$?

Yes to the first. Check the second.

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) dx dy &= \int_0^1 \int_0^1 1 dx dy \\ &= \int_0^1 [x_0^1] dy \\ &= \int_0^1 1 dy \\ &= y|_0^1 \\ &= 1 \end{aligned}$$

So yes, it is a joint density function.

Cumulative density functions

Using the basic definition of a density function

$$\begin{aligned}
\Pr(x \leq b \text{ and } y \leq d) &= \Pr(-\infty \leq x \leq b \text{ and } -\infty \leq y \leq d) \\
&= \int_{-\infty}^d \int_{-\infty}^b f(x,y) dx dy \\
&= F(b,d)
\end{aligned}$$

where $F(b,d)$ is the joint cumulative density function evaluated at b and d . Note that $F(b,d)$ is defined for all b and d , $-\infty < b,d < \infty$. So, $F(x,y) = \Pr(X \leq x \text{ and } Y \leq y)$

An example:

Consider the density

$$f(x,y) = \begin{cases} x+y & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $0 \leq b \leq 1$ and $0 \leq d \leq 1$

$$\begin{aligned}
F(b,d) &= \int_0^d \int_0^b (x+y) dx dy = \int_0^d [.5x^2 + xy]_0^b dy \\
&= \int_0^d (.5b^2 + by) dy \\
&= .5b^2y + .5by^2 \Big|_0^d \\
&= .5b^2d + .5bd^2 \\
&= .5bd(b+d)
\end{aligned}$$

Therefore, (this is a bit tricky)

$$F(b,d) = \begin{cases} 0 & \text{if } b < 0 \text{ or } d < 0 \\ .5bd(b+d) & \text{if } 0 \leq b < 1 \text{ and } 0 \leq d \leq 1 \\ 1 & \text{if } b > 1 \text{ and } d > 1 \\ .5b(b+1) & \text{if } 0 \leq b \leq 1 \text{ and } d > 1 \\ .5d(1+d) & \text{if } b > 1 \text{ and } 0 \leq d \leq 1 \end{cases}$$

So, for example,

$$F(.5, .25) = .5(.5)(.25)(.5 + .25) = 4.6875 \times 10^{-2} = \frac{3}{64}$$

which is the probability that $x \leq .5$ and $y \leq .25$. It is also the volume under the surface $z = x + y$ over the region $\{(x,y) : 0 \leq x \leq .5, 0 \leq y \leq .25\}$. I think you will find this example of a joint density function on page 139 of MGB

Example 2:

If

$$f(x,y) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

In this example, if $0 \leq b \leq 1$ and $0 \leq d \leq 1$

$$\begin{aligned} F(b,d) &= \int_0^d \int_0^b 1 dx dy = \int_0^d [x|_0^b] dy \\ &= \int_0^d b dy \\ &= by|_0^d \\ &= bd \end{aligned}$$

Therefore

$$F(b,d) = \begin{cases} 0 & \text{if } b < 0 \text{ or } d < 0 \\ bd & \text{if } 0 \leq b < 1 \text{ and } 0 \leq d \leq 1 \\ 1 & \text{if } b > 1 \text{ and } d > 1 \\ b & \text{if } 0 \leq b \leq 1 \text{ and } d > 1 \\ d & \text{if } b > 1 \text{ and } 0 \leq d \leq 1 \end{cases}$$

So, for example,

$$F(.5, .5) = .5(.5) = .25$$

which is the probability that $x \leq .5$ and $y \leq .5$. It is also the volume under the surface $f(x,y)$ over the region $\{(x,y) : -\infty \leq x \leq .5, -\infty \leq y \leq .5\}$

Marginal density functions

Consider the joint density function $f(x,y)$. If

$$\Pr(a \leq x \leq b \text{ and } c \leq y \leq d) = \int_a^b \int_c^d f(x,y) dy dx$$

then what is the probability that $a < x < b$? It is

$$\Pr(a \leq x \leq b \text{ and } -\infty \leq y \leq +\infty) = \int_a^b \int_{-\infty}^{+\infty} f(x,y) dy dx$$

If we define $f_x(x) \equiv \int_{-\infty}^{+\infty} f(x,y) dy$, then $\Pr(a \leq x \leq b \text{ and } -\infty \leq y \leq +\infty)$ can be rewritten as

$$\begin{aligned} & \Pr(a \leq x \leq b \text{ and } -\infty \leq y \leq +\infty) \\ &= \Pr(a \leq x \leq b) = \int_a^b f_x(x) dx \end{aligned}$$

where $f_x(x)$ is called the marginal density function of the random variable x .

Are you sure $f_x(x)$ is a density function. For $f_x(x)$ to be a density function it must be the case that $f_x(x) \geq 0 \forall x$ and $\int_{-\infty}^{+\infty} f_x(x) dx = 1$. Check that it is

$$\begin{aligned} & \int_{-\infty}^{+\infty} f_x(x) dx \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) dy dx \text{ by definition of } f_x(x) \\ &= 1 \text{ because } f(x,y) \text{ was defined as the joint density function} \end{aligned}$$

Calling $f_x(x)$ the marginal density function emphasizes that x is jointly distributed with other variables (in our simple case, y)

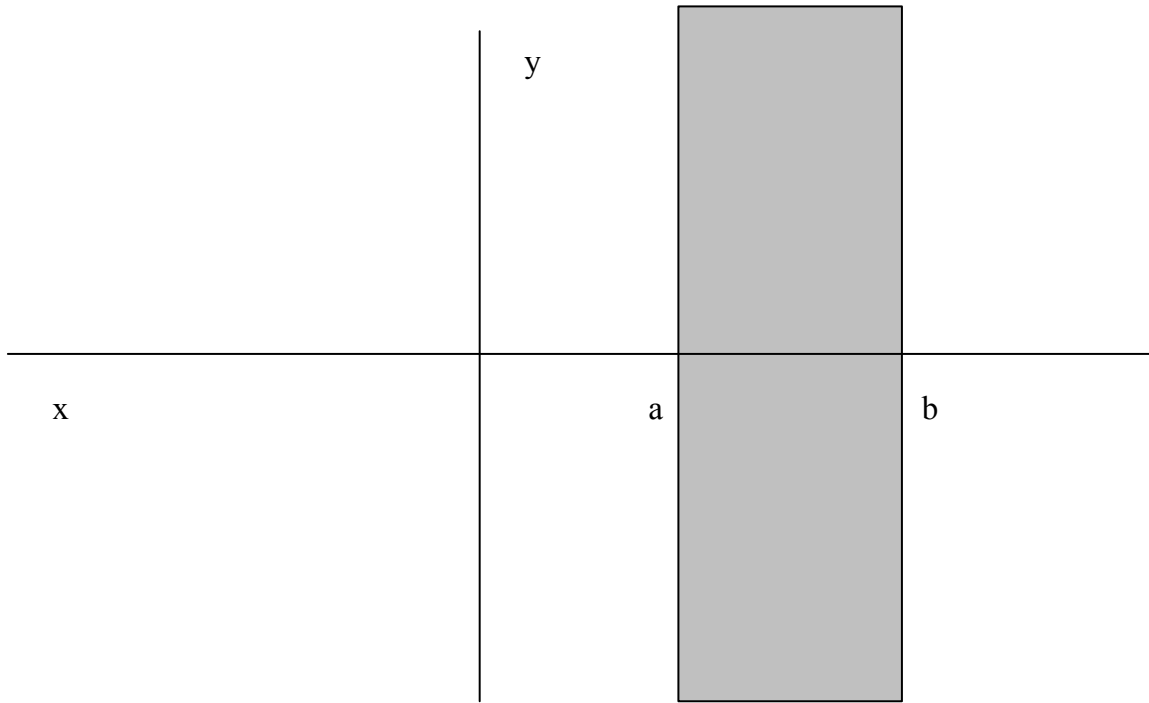
Alternatively,

$$\begin{aligned} \Pr(c \leq y \leq d) &= \Pr(-\infty \leq x \leq +\infty \text{ and } c \leq y \leq d) \\ &= \int_c^d \int_{-\infty}^{+\infty} f(x,y) dx dy = \int_c^d f_y(y) dy \end{aligned}$$

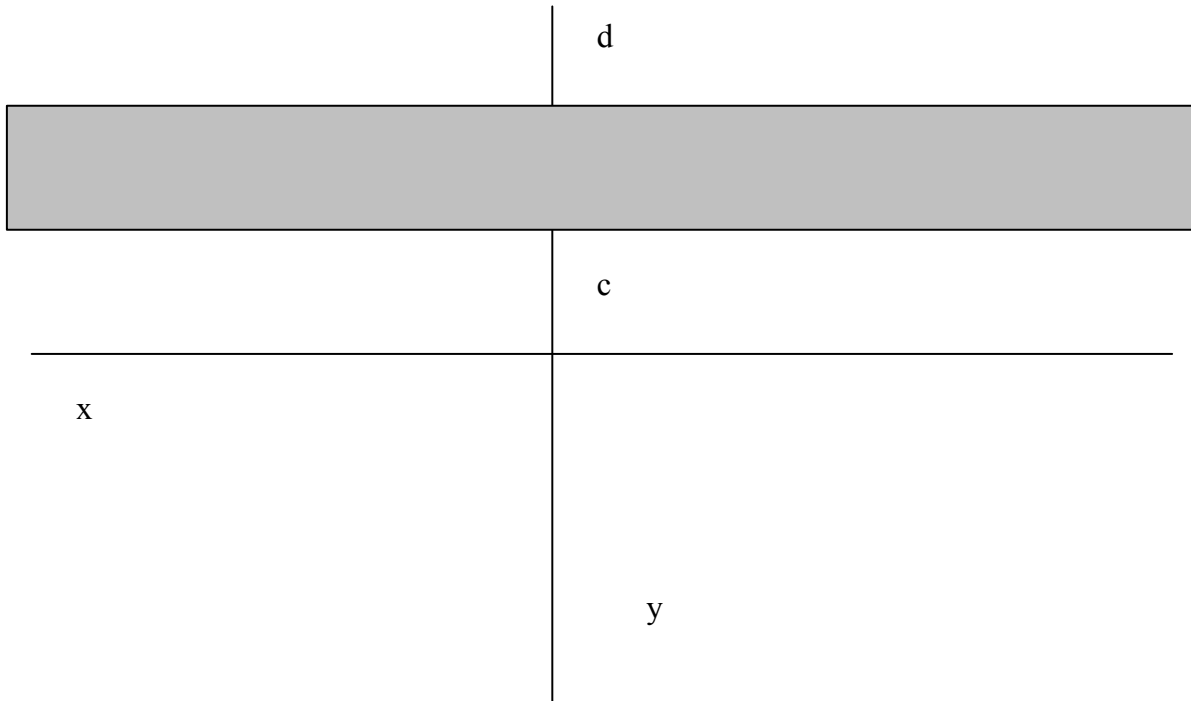
where $f_y(y)$ is the marginal density function of y . That is

$$f_y(y) = \int_{-\infty}^{+\infty} f(x,y) dx$$

Graphically, $\Pr(a < x < b)$ is the volume under $f(x,y)$ above the shaded area



and, $\Pr(c < y < d)$ is the volume under $f(x,y)$ above the shaded area



Now consider some conditional density functions

If x and y are jointly distributed with density function $f(x, y)$, what is the probability that $(a < x < b)$ given that $y = c$? Define

$$\Pr(a < x < b | y = c) \equiv \int_a^b f(x | y = c) dx$$

where $f(x | y = c)$ is the density of x given that $y = c$. $f(x | y = c)$ is called a conditional density function. Let's try and identify $f(x | y = c)$ and see how it relates to the functions $f(x, y)$ and $f(x, c)$.

One might first guess that $f(x | y = c) = f(x, c)$, but this is not the case because $f(x, c)$ is not a density function: the area under it is not equal to one. To make $f(x, c)$ a density function, one needs to scale it (adjust it) so that the area under it is equal to 1. Consider the possibility that

$$f(x | y = c) = \frac{f(x, c)}{f_y(c)}$$

By definition of $f_y(y)$,

$$f(x|y = c) = \frac{f(x, c)}{\int_{-\infty}^{+\infty} f(x, c) dx}$$

In which case

$$\int_{-\infty}^{+\infty} f(x|y = c) dx = \int_{-\infty}^{+\infty} \frac{f(x, c)}{\int_{-\infty}^{+\infty} f(x, c) dx} dx = \frac{\int_{-\infty}^{+\infty} f(x, c) dx}{\int_{-\infty}^{+\infty} f(x, c) dx}$$

because the original $\int_{-\infty}^{+\infty} f(x, c) dx$ is a constant

$$= 1$$

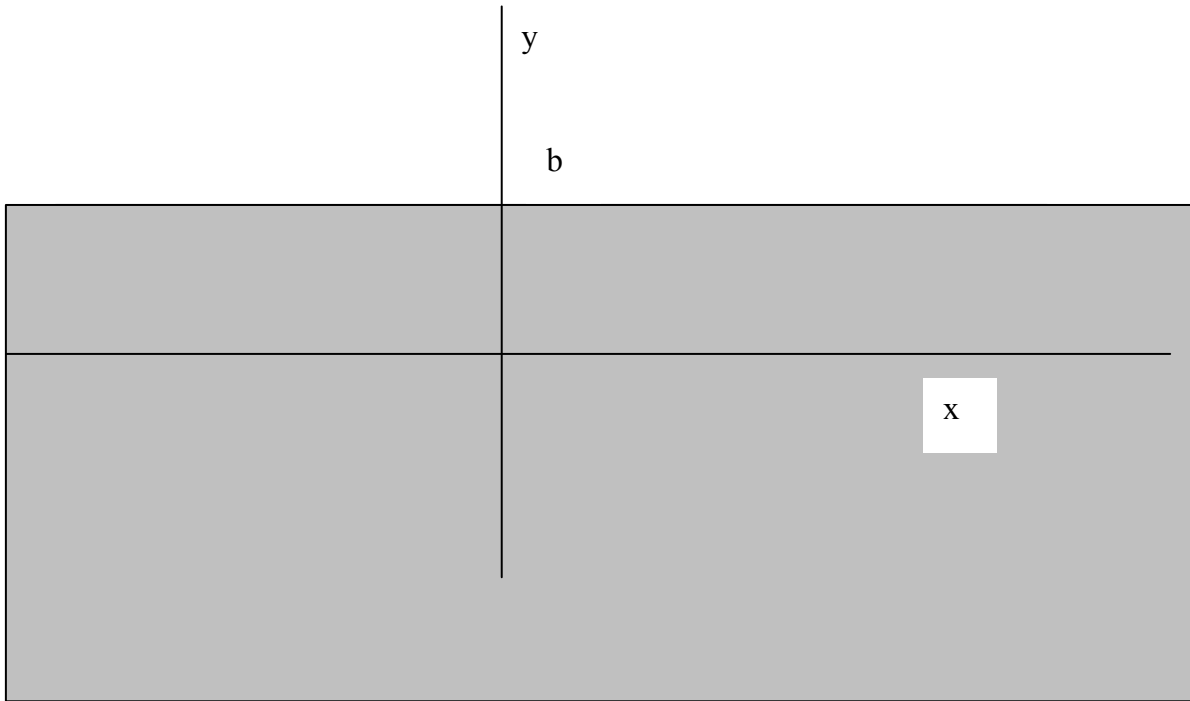
so $f(x|y = c) = \frac{f(x, c)}{f_y(c)}$ is a legitimate density function. Convince yourself that it can be used to correctly estimate the $\Pr(a < x < b | y = c)$. $f(x|y = c)$ is often written more loosely as $f(x|y) = \frac{f(x, y)}{f_y(y)}$.

Let's get a bit more complicated with conditional densities. Consider the more complicated conditional density function ; that is, the density of x conditional on $y \leq b$. It can be shown that

$$f(x|y \leq b) = \frac{\int_{-\infty}^b f(x, y) dy}{\int_{-\infty}^{+\infty} \int_{-\infty}^b f(x, y) dy dx}$$

$$= \frac{\int_{-\infty}^b f(x, y) dy}{\Pr(y \leq b)}$$

The denominator, $\Pr(y \leq b)$ is the volume above the shaded area.

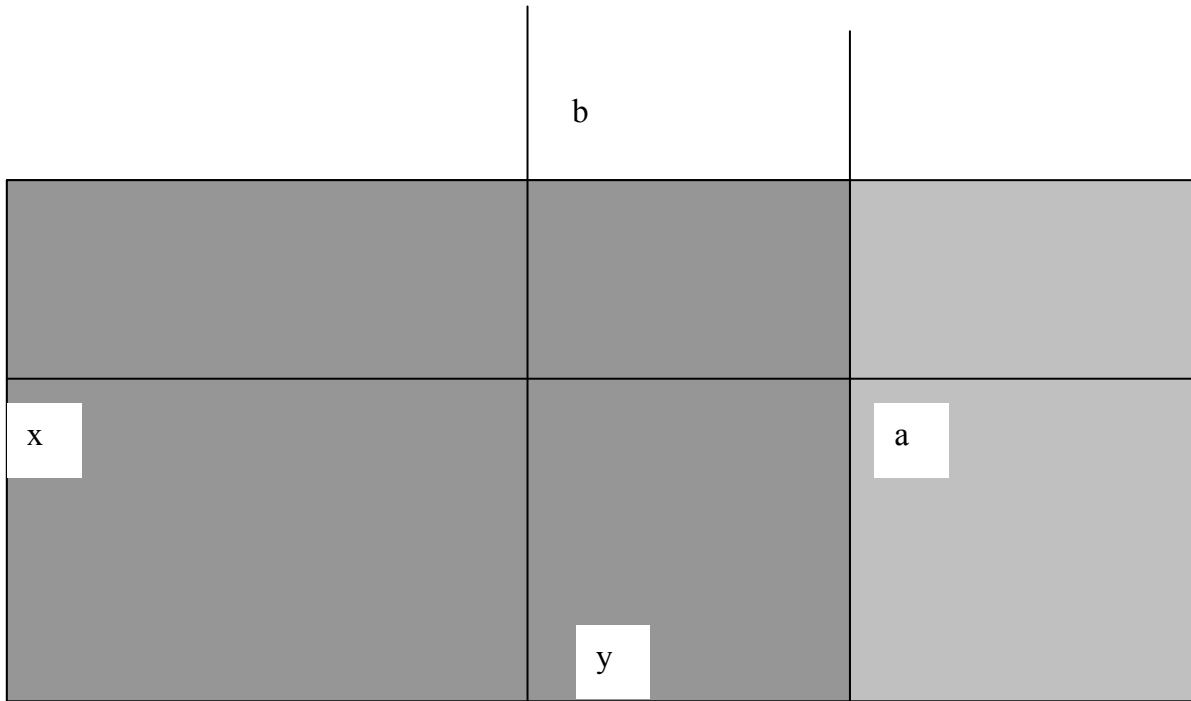


Now consider the probability that $(x \leq a | y \leq b)$.

$$\Pr(x \leq a | y \leq b) = \int_{-\infty}^{+\infty} f(x | y \leq b) dx$$

$$= \frac{\int_{-\infty}^a \int_{-\infty}^b f(x, y) dy}{\int_{-\infty}^{+\infty} \int_{-\infty}^b f(x, y) dy dx}$$

The denominator is the volume above the two shaded areas and the numerator is the volume above the darker shaded region.



Expected values

If x and y are jointly distributed with density function $f(x,y)$,

$$E[g(x,y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y)f(x,y)dx dy$$

Begin by considering the special case $g(x,y) = x$. In which case,

$$\begin{aligned} E[x] &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xf(x,y)dx dy \\ &= \int_{-\infty}^{+\infty} x \int_{-\infty}^{+\infty} f(x,y)dy dx \\ &= \int_{-\infty}^{+\infty} xf_x(x)dx \end{aligned}$$

and by analogy $E[y] = \int_{-\infty}^{+\infty} yf_y(y)dy$.

Now consider

$$g(x,y) = (x - E[x])(y - E[y])$$

So

$$E[(x - E[x])(y - E[y])] \\ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [(x - E[x])(y - E[y])]f(x,y)dxdy$$

This number is called the covariance between the random variables x and y . $cov(x,y)$ is a measure of the **linear** relationship between x and y . Note that

$$cov(x,y) \leq 0$$

if $cov(x,y) = 0$, x and y are uncorrelated. If $cov(x,y) > 0$, x and y are positively correlated. If $cov(x,y) < 0$, x and y are negatively correlated. Note that zero covariance is not the same thing as independence.

Can you show me that

$$cov(x,y) = E[xy] - E[x]E[y]$$

proof

$$cov(x,y) = E[(x - E[x])(y - E[y])] \\ = E[xy - xE[y] - E[x]y + E[x]E[y]] \\ = E[xy] - E[x]E[y] - E[x]E[y] + E[x]E[y] \\ = E[xy] - E[x]E[y]$$

So, if $cov(x,y) = 0$, $E[xy] = E[x]E[y]$.

$cov(x,y)$ can take almost any value and it is not possible to tell whether the correlation between x and y is high or low by observing the covariance. To "fix" this we sometimes

rescale it so it falls in the -1 to $+1$ interval. This is accomplished by dividing it by $\sigma_x\sigma_y$ where $\sigma_x^2 \equiv E[(x - E[x])^2]$. That is,

$$0 \leq \rho \equiv \frac{cov(x,y)}{\sigma_x\sigma_y} \leq 1$$

ρ is called the correlation coefficient.

Independence

If $f(x,y) = f_x(x)f_y(y)$, then x and y are defined as being statistically independent. That is, if the joint density function can be written as the product of the marginal density functions of its random variables, those random variables are statistically independent.

Note that the independence condition $f(x,y) = f_x(x)f_y(y)$ can be rearranged to $f_x(x) = \frac{f(x,y)}{f_y(y)}$ or $f_y(y) = \frac{f(x,y)}{f_x(x)}$. And, since $f(x|y) = \frac{f(x,y)}{f_y(y)}$ is a conditional density function,

$$\text{independence of } x \text{ and } y \Leftrightarrow f_x(x) = f(x|y) \Leftrightarrow f_y(y) = f(y|x)$$

In words, if the random variables x and y are independent, the conditional distribution for each variable is the same as that variable's marginal distribution. If x and y are independent, the

distribution of x does not depend on y and vice versa. Independence is a strong condition.

It can be shown (MGB p150) that

$$f(x,y) = f_x(x)f_y(y) \Leftrightarrow F(x,y) = F_x(x)F_y(y)$$

So, independence is equivalent to the joint CDF being the product of the CDFs for each of its random variables.

If x and y are independent, one can show that (MGB 160 with proof) that

$$E[g_1(x)g_2(y)] = E[g_1(x)]E[g_2(y)]$$

Recollect that uncorrelated is not the same thing as independent. Simply put

$$x \text{ and } y \text{ independent} \Rightarrow \text{cov}(x,y) = 0$$

but

$$\text{cov}(x,y) = 0 \not\Rightarrow x \text{ and } y \text{ independent}$$

Independence tells us that x and y do not vary together in any way. $\text{cov}(x,y) = 0$ tells us that there is no linear relationship between x and y . MGB have an example on page 161 where x and y are uncorrelated, but not independent.