The profit function

April 2, 2001

Consider a competitive firm with the production function

$$y = f(k,l)$$

Assume proft (π) maximizing behavior.

$$\pi = py - wl - rk$$

where $y = f(k,l)$, w is the price of labor, r is the price of captial, and p is the price of y. Substituting the constraint into the objective function

$$\pi = pf(k,l) - wl - rk$$

One maximizes this wrt k and l to get, the input demand functions, $k^* = k(p,w,r)$ and $l^* = l(p,w,r)$. That is, (k^*,l^*) is the vector of inputs that maximizes profts given p, w, and r.

Make sure you understand the distinction between the firm's input demand functions and its conditional input demand functions.

Note that $y^* = f(k^*,l^*) = f(k(p,w,r),k(p,w,r)) \ \ = g(p,w,r)$ is the firm's supply function

If $\pi = py - wl - rk$, then max π, π^*

$$\begin{aligned}\pi^* &= pg(p,w,r) - wl(p,w,r) - rl(p,w,r) \\ &= \pi(p,w,r)\end{aligned}$$

$\pi(p,w,r)$ is called the profit function. It identifies maximum profits as a function of p, w, and r.

Profit functions are another way to characterize the technology for producing y. That is,

$$f(k,l) \text{ and } w,r,p \Rightarrow \pi(p,w,r)$$

and

$$f(k,l) \text{ and } w,r,p \Leftarrow \pi(p,w,r)$$

Therefore,

$$f(k,l) \text{ and } w,r,p \Leftrightarrow \pi(p,w,r)$$

Now let's generalize to a firm with N inputs. To simplify the notation, let $\mathbf{w}$ denote the vector of input prices. In which case the profit function can be written $\pi(p, \mathbf{w})$. All profit functions have the following properties

1. nondecreasing in p
2. nonincreasing in $\mathbf{w}$
3. homogenous of degree one in p and $\mathbf{w}$
4. convex in p and $\mathbf{w}$
5. continuous in p and $\mathbf{w}$

That is, if a function of p and $\mathbf{w}$ has these five properties it is a profit function. If not, it is not a profit function.

Now for the good part

Hotelling's lemma

It can be shown that

$$y^* = y(p, \mathbf{w}) = \frac{\partial \pi(p, \mathbf{w})}{\partial p}$$

and

$$x_i^* = x_i(p, \mathbf{w}) = -\frac{\partial \pi(p, \mathbf{w})}{\partial w_i}$$

where x_i is the quantity of input i. Given the profit function, it is very simple to derive the firm's supply and demand functions.

An example profit function

$$\pi(p, \mathbf{w}) = p^{-(\frac{1}{\beta-1})}(w_1^\gamma + w_2^\gamma)^{(\frac{\beta}{\gamma(\beta-1)})}\beta^{-(\frac{\beta}{\beta-1})}(1-\beta)$$

where $\gamma = \frac{\rho}{(\rho-1)}$ and $0 < \beta < 1$. Derive the demand and supply functions corresponding to this profit function. Deriving the supply function

$$\begin{aligned}
& y^s(p, \mathbf{w}) \\
&= {}^{-(\frac{1}{\beta-1})}\, p^{-(\frac{1}{\beta-1})-1}(w_1^\gamma + w_2^\gamma)^{(\frac{\beta}{\gamma(\beta-1)})}\beta^{-(\frac{\beta}{\beta-1})}(1-\beta) \\
&= {}^{-(\frac{1}{\beta-1})}\, p^{-(\frac{\beta}{\beta-1})}(w_1^\gamma + w_2^\gamma)^{(\frac{\beta}{\gamma(\beta-1)})}\beta^{-(\frac{\beta}{\beta-1})}(1-\beta) \\
&= p^{-(\frac{\beta}{\beta-1})}(w_1^\gamma + w_2^\gamma)^{(\frac{\beta}{\gamma(\beta-1)})}\beta^{-(\frac{\beta}{\beta-1})}
\end{aligned}$$

Can you find the production function associated with this profit function. It is possible but will take a lot of algebra.

Derive a profit function. For example, start with the simple CD production function $y = f(k,l) = k^{.4}l^{.4}$ and derive the profit function. Then apply Hotelling's lemma.

An alternative derivation of the profit function

Consider the profit maximization problem in terms of the cost function rather than the production function

$$\pi = py - c(y, \mathbf{w})$$

Maximizing this wrt y, one obtains the supply function

$$y^* = y(p, \mathbf{w})$$

Plug this inte π to get

$$\begin{aligned} \pi &= py(p, \mathbf{w}) - c(y, \mathbf{w}) \\ &= \pi(p, \mathbf{w}) \end{aligned}$$

Again we have the profit function.

When one starts the theory of the competitive firm, one can start with the profit function.

Note the following:

given the profit function $\pi(p, \mathbf{w})$ and Hotelling's lemma

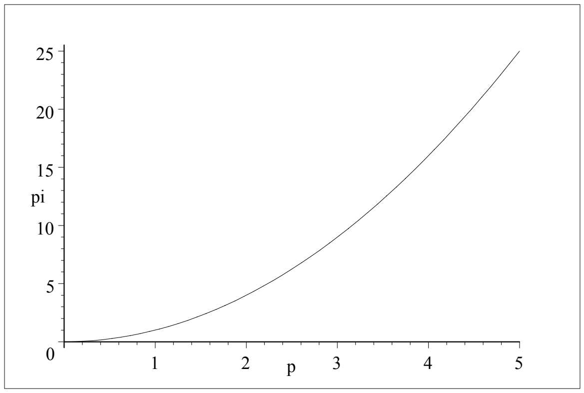
$$y^s = \frac{\partial \pi(p, \mathbf{w})}{\partial p}$$

so

$$\frac{\partial y^s}{\partial p} = \frac{\partial^2 \pi(p, \mathbf{w})}{\partial p^2}$$

That is, the slope of the supply function is $\frac{\partial^2 \pi(p, \mathbf{w})}{\partial p^2}$. Can you use the properties of the profit function to prove that the supply function for a competitive firm cannot slope down?

Yes - if $(p, \mathbf{w})$ is convex in p and nondecreasing in p then $\frac{\partial y^s}{\partial p} = \frac{\partial^2 \pi(p,\mathbf{w})}{\partial p^2} \geq 0$. If strictly increasing in p then $\frac{\partial y^s}{\partial p} = \frac{\partial^2 \pi(p,\mathbf{w})}{\partial p^2} > 0$, and



convex and increasing in p

Can you prove something comparable wrt the input demand function for the competitive firm?

The demand function for input i is $x_i^* = -\frac{\partial \pi(p,\mathbf{w})}{\partial w_i}$ by Hotelling's lemma. There the slope of the demand function

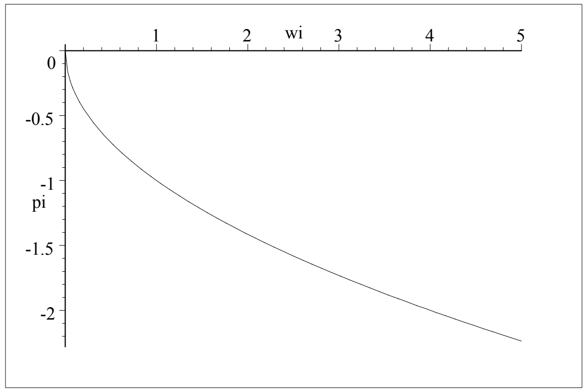
$$\frac{\partial x_i}{\partial w_i} = \frac{\partial(-\frac{\partial \pi(p,\mathbf{w})}{\partial w_i})}{\partial w_i} = -\frac{\partial^2 \pi(p,\mathbf{w})}{\partial {w_i}^2}$$

This must be nonpositive because the profit function in nonincreasing and convex in w_i. Nondecreasing and convex implies that $\frac{\partial^2 \pi(p,\mathbf{w})}{\partial {w_i}^2} \geq 0$, so

$$\frac{\partial x_i}{\partial w_i} = -\frac{\partial^2 \pi(p,\mathbf{w})}{\partial {w_i}^2} \leq 0$$

That is, input demand functions do not slope upward. Consider the special case where the profit function in strictly decreasing and strictly convex in w_i. The graph of π as a function w_i looks something like

$-x^{.5}$



strictly decreasing and strictly convex in wi