## Now let's briefly consider multiple integrals.

For example,

assume the function

$f(x, y)$

What is the form of

$\int\int f(x, y)\, dy\, dx$? (Note that $\int\int f(x, y)\, dy\, dx = \int\int f(x, y)\, dx\, dy$)

That is, what is the form of the function

$G(x, y)$

such that

$G_{xy}(x, y) = G_{yx}(y, x) = f(x, y)$

This is double integration: f(x,y) is first integrated wrt y, and the result is then integrated wrt x.

Let's start with an example

One can think of multiple integrals as follows:

$\iint f(x,y)dydx = \int m(x,y)dx$ , where

$m(x,y) = \int f(x,y)dy$

where

$G(x, y) = \frac{3}{4} x^2 y^2 + 2x^2 + y^2 + 4$

| Note: 4 terms |
|---|
| One a function of x and y<br>One a function of x<br>One a function of y<br>One a function of neither x nor y |

Given this $G(x, y)$

what is $f(x, y) = G_{yx}(y, x) = G_{xy}(x, y)$?

$G_y = 1.5x^2y + 2y$ and $G_x = 1.5xy^2 + 4x$

and

$G_{yx} = G_{xy} = 3xy$; that is, $f(x, y) = 3xy$

Try to recover $G(x, y)$ from $f(x, y) = 3xy$

start with

(1) $\int f(x, y)\, dy = \int G_{xy}(x, y)\, dy = \int (3xy)\, dy = 1.5xy^2 + \alpha'(x) = G_x(x, y)$

Note: the constant of integration, $\alpha'(x) \equiv d\alpha(x)/dx$ , can depend on x. I could have given this constant of integration any name. In a moment you will see why I wanted to express it as a derivative of a function. Note that $\alpha'(x)$ is not a function of $y$ because if there was a term that is a function of $y$ but not $x$ it would have appeared.

(2) $\int f(x,y)\,dx = \int G_{yx}(x,y)dx = \int (3xy)\,dx = 1.5x^2y + \beta'(y) = G_y(x,y)$

(note: the constant of integration, $B'(y) \equiv dB(y)/dy$ , can depend on y)

Note that $\alpha'(x) = 4x$ but we don't know this

and

$\beta'(y) = 2y$ but we don't know this

now integrate (1) wrt x

$$\int\int f(x,y)\,dy\,dx = \int G_x(x,y)dx = \int (1.5xy^2 + \alpha'(x))\,dx = \underbrace{\tfrac{3}{4}x^2y^2}_{\text{integral of } 1.5xy^2 \text{ wrt x}} + \beta(y) + \gamma_1 + \alpha(x)$$

This is G(x,y), except we don't know functional forms of $\alpha(x)$, $\beta(y)$, $\gamma_1$.

Alternatively integrate (2) wrt y

$$\int\int f(x,y)\,dx\,dy = \int G_y(x,y)dy = \int (1.5x^2y + \beta'(y))\,dy = \underbrace{\tfrac{3}{4}x^2y^2}_{\text{integral of } 1.5x^2y \text{ wrt y}} + \alpha(x) + \gamma_2 + \beta(y)$$

Again, this is G(x,y), but we don't know the functional forms of $\alpha(x)$, $\beta(y)$ or the values of the constants $\gamma_1$ and $\gamma_2$.

But since

$$\int\int (3xy)\,dx\,dy = \int\int (3xy)\,dy\,dx \Rightarrow \gamma_1 = \gamma_2$$

$$\int\int (3xy)\,dx\,dy$$

$$= \tfrac{3}{4}x^2y^2 + \alpha(x) + \beta(y) + \gamma$$

Reviewing, if

$$G(x,y) = \tfrac{3}{4}x^2y^2 + \alpha(x) + \beta(y) + \gamma$$

but start with

$$G_{xy} = G_{yx} = f(x, y)$$

can only explicitly recover

$$\tfrac{3}{4}x^2y^2$$

which would = $G(x, y)$
if $\alpha(x) = \beta(y) = \gamma = 0$,
but not otherwise.

One can't recover the forms of $\alpha(x)$, $\beta(y)$, or $\gamma$ from just $f(x, y)$;
i.e. we don't know that $\alpha(x) = 2x^2$, $\beta(y) = y^2$, or $\gamma = 4$.

So, in general, can't recover $G(x, y)$ from knowledge of just $G_{xy}(x, y)$ by integrating

$$\int\int G_{xy}(x, y)\, dy\, dx$$

one can't recover all the terms.

How about with knowledge of $G_x(x, y)$, $G_y(x, y)$ or both?

Let's investigate this issue in terms of our specific example

$$G(x, y) = \tfrac{3}{4}x^2y^2 + 2x^2 + y^2 + 4$$

$$G_x = 1.5xy^2 + 4x$$

$$G_y = 1.5x^2y + 2y.$$

Integrate $G_x$ wrt x

(1) $\int G_x(x, y)\, dx = \int (1.5xy^2 + 4x)\, dx = \tfrac{3}{4}x^2y^2 + \beta(y) + 2x^2 + \gamma_1$

(1a) so $G(x, y) = \tfrac{3}{4}x^2y^2 + \beta(y) + 2x^2 + \gamma_1$

(2) Now integrate $G_y(x, y)$ wrt y

$$\int G_y(x, y)\, dy = \int (1.5x^2y + 2y)\, dy = \tfrac{3}{4}x^2y^2 + \alpha(x) + y^2 + \gamma_2$$

(2a) so $G(x, y) = \tfrac{3}{4}x^2y^2 + \alpha(x) + y^2 + \gamma_2$

We can't recover $\beta(y)$ from (1a) and we can't recover $\alpha(x)$ from (1b), but we can recover $\alpha(x)$ and $\beta(y)$ by noting that

(1a) = (2a) $\Rightarrow$

$$G(x, y) = \tfrac{3}{4}x^2y^2 + 2x^2 + y^2 + \gamma$$

i.e. from the <u>two</u> partial derivatives of G(x, y), $G_x(x, y)$ and $G_y(x, y)$, we can recover G(x, y) up to the constant, $\gamma$.

In general, if there is a function G(x, y) that we do not observe, but if we know <u>both</u> $G_x(x, y)$ and $G_y(x, y)$, we can recover G(x, y) up to an unknown constant

by integrating $G_x(x, y)$ wrt x

by integrating $G_y(x, y)$ wrt y

and noting

$$G(x, y) = \int G_x \, dx = \int G_y \, dy$$

**Now let's briefly consider indefinite multiple integrals.**

Practice Quiz

Assume that there exists some production function

$$x = f(K, L)$$

where

the marginal product of labor is

$$(1) \qquad MP_L \equiv \frac{\partial f(K,L)}{\partial L} = .125\, K^2 L^{-.5} + 6L$$

and

$$(2) \qquad MP_K \equiv \frac{\partial f(K,L)}{\partial K} = .5\, KL^{.5} + 3K^{.5}$$

Derive the production function, f(K, L)

$$f(K,L) = \int MP_L \, dL \equiv \int (.125\, K^2 L^{-.5} + 6L)\, dL$$

$$(1a) \qquad = \underbrace{.25 K^2 L^{.5} + \alpha(K)}_{\text{integral of } .125K^2L^{-5} \text{ wrt L}} + 3L^2 + \gamma_1$$

and

$$f(K,L) = \int MP_K \, dK \equiv \int (.5 KL^{.5} + 3K^{.5})\, dK$$

$$(1b) \qquad = \underbrace{.25 K^2 L^{.5} + \beta(L)}_{\text{integral of } .5KL^{.5} \text{ wrt K}} + 2K^{1.5} + \gamma_2$$

Since (1a) = (1b)

$$f(K,L) = .25 K^2 L^{.5} + 3L^2 + 2K^{1.5} + \gamma$$

i.e. we have recovered f(K, L) except for the unknown constant, $\gamma$.

Can we determine $\gamma$?

For production functions, physics tells us

Zero in $\Rightarrow$ Zero out

i.e.

$$f(0, 0) = 0$$

so $\gamma = 0$ and

$$f(K,L)\ .25 K^2 L^{.5} + 3L^2 + 2K^{1.5}$$

Example 2

Assume some cost function, c(x, w, r) which might or might not be a long-run cost function, where

marginal cost, $Mc_x$, is

$$MC_x = \frac{\partial c(x, w, r)}{\partial x} = .5x^{-.5} w^{.3} r^{.75}$$

the conditional demand function for labor, $L_c^d(x, w, r)$, is

$$L_c^d = \frac{\partial c(x, w, r)}{\partial w} = .3x^{.5} w^{-.7} r^{.75}$$

and the conditional demand function for capital, $K_c^d(x, w, r)$, is

$$K_d^c = \frac{\partial c(x, w, r)}{\partial r} = .75x^{.5} w^{.3} r^{-.25} + 4$$

Derive, if possible, c(x, w, r) **using integration**[1]

(1a)

$$c(x, w, r) = \int MC_x \, dx = \int (.5x^{-.5} w^{.3} r^{.75}) \, dx$$

$$= x^{.5} w^{.3} r^{.75} + \alpha(w, r) + \beta(w) + \gamma(r) + \eta$$

Note that the constant of integration can be a function of w and r, so can include as many as four terms:

and

(2a)

$$c(x, w, r) = \int L_c^d \, dw = \int (.3x^{.5} w^{-.7} r^{.75}) \, dw$$

$$= x^{.5} w^{.3} r^{.75} + \chi(x, r) + \psi(x) + \gamma(r) + \eta$$

and

(3a) $$c(x, w, r) = \int K_c^d \, dr = \int (.75x^{.5} w^{.3} r^{-.25} + 4) \, dr$$

$$= x^{.5} w^{.3} r^{.75} + \phi(x, w) + \psi(x) + 4r + \beta(w) + \eta$$

[1]Not that this is a silly way of deriving the cost function, given that we have the two conditional demand functions, since each weighted by the respective input price is the cost function by definition; that is, $c(x, w, r) = wL_c^d(x, w, r) + rK_c^d(x, w, r)$

(1a) ⇒ $\psi(x) = 0$ (if $\psi(x) \neq 0$, there would be a term in (1a) that is a function of only x)

(2a) ⇒ $\beta(w) = 0$ (if $\beta(w) \neq 0$, there would be a term in (2a) that is a function of only w)

(3a) ⇒ $\gamma(r) = 4r$ (the term that is just a function of r is 4r)

(2a) or (3a) $\Rightarrow \alpha(w,r) = 0$, otherwise there be a term in (2a) and (3a) that was solely a function of w and r.

(1a) $\Rightarrow \chi(x,r) = 0$ , otherwise there would be a term in (1a) that was solely a function of x and r.

(1a) $\Rightarrow \varphi(x,w) = 0$ , otherwise there would be a term in (1a) that was solely a function of x and w.

Therefore,

(1b) $c(x, w, r) = x^{.5} w^{.3} r^{.75} + 4r + \eta$

(2b) $c(x, w, r) = x^{.5} w^{.3} r^{.75} + 4r + \eta$

and

(3b) $c(x, w, r) = x^{.5} w^{.3} r^{.75} + 4r + \eta$

Since (1b) = (2b) = (3b)

$$c(x, w, r) = x^{.5} w^{.3} r^{.75} + 4r + \eta$$

but, if there are only 2 inputs K & L with prices w & r, it must be the case that

$$c(x, 0, 0) = 0 \quad \Rightarrow \quad \eta = 0$$

So

$$c(x, w, r) = x^{.5} w^{.3} r^{.75} + 4r.$$

Wow!

Is it a longrun or shortrun cost function?

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